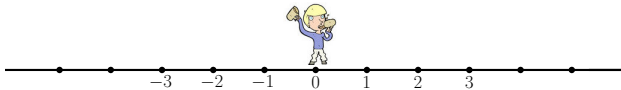


A lost man will reach home, but a lost bird will be
lost forever

Perla Sousi ¹

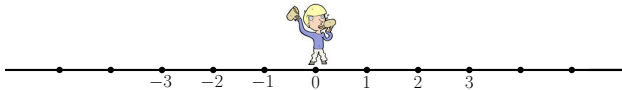
¹Emmanuel College, University of Cambridge

Random walk on \mathbb{Z}



At time 0 he starts from home.

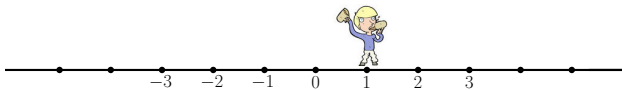
Random walk on \mathbb{Z}



At time 0 he starts from home.

He tosses a fair coin. If **H**, he takes a step to the right. If **T**, a step to the left.

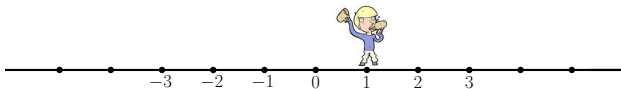
Random walk on \mathbb{Z}



At time 0 he starts from home.

He tosses a fair coin. If **H**, he takes a step to the right. If **T**, a step to the left.

Random walk on \mathbb{Z}

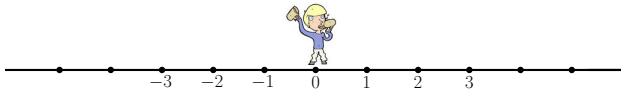


At time 0 he starts from home.

He tosses a fair coin. If **H**, he takes a step to the right. If **T**, a step to the left.

He continues in the same way by tossing independent fair coins.

Random walk on \mathbb{Z}

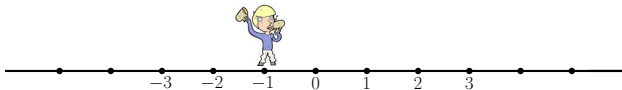


At time 0 he starts from home.

He tosses a fair coin. If **H**, he takes a step to the right. If **T**, a step to the left.

He continues in the same way by tossing independent fair coins.

Random walk on \mathbb{Z}

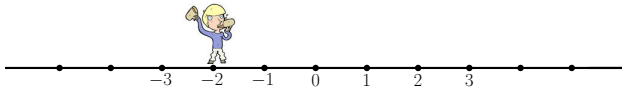


At time 0 he starts from home.

He tosses a fair coin. If **H**, he takes a step to the right. If **T**, a step to the left.

He continues in the same way by tossing independent fair coins.

Random walk on \mathbb{Z}

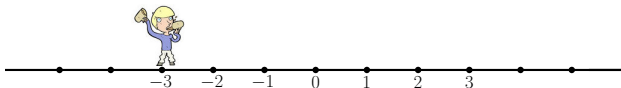


At time 0 he starts from home.

He tosses a fair coin. If **H**, he takes a step to the right. If **T**, a step to the left.

He continues in the same way by tossing independent fair coins.

Random walk on \mathbb{Z}

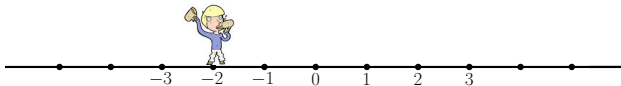


At time 0 he starts from home.

He tosses a fair coin. If **H**, he takes a step to the right. If **T**, a step to the left.

He continues in the same way by tossing independent fair coins.

Random walk on \mathbb{Z}

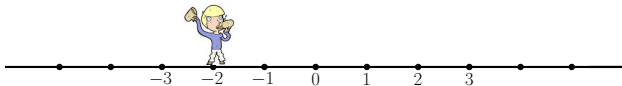


At time 0 he starts from home.

He tosses a fair coin. If **H**, he takes a step to the right. If **T**, a step to the left.

He continues in the same way by tossing independent fair coins.

Random walk on \mathbb{Z}



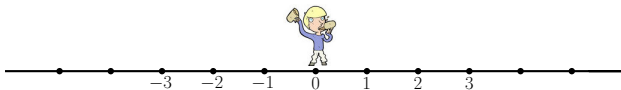
At time 0 he starts from home.

He tosses a fair coin. If **H**, he takes a step to the right. If **T**, a step to the left.

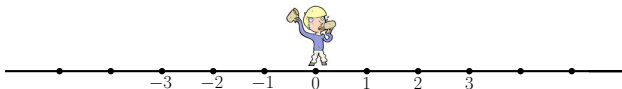
He continues in the same way by tossing independent fair coins.

This is called a **simple symmetric random walk on \mathbb{Z}** .

Random walk on \mathbb{Z}



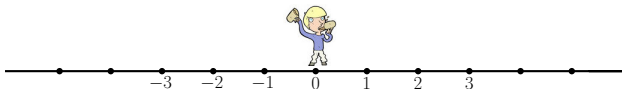
Random walk on \mathbb{Z}



Question

Will he ever get back home (*return to 0*)?

Random walk on \mathbb{Z}

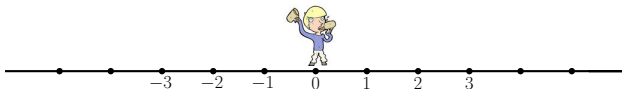


Question

Will he ever get back home (*return to 0*)?

Set p = probability of ever returning to 0.

Random walk on \mathbb{Z}



Question

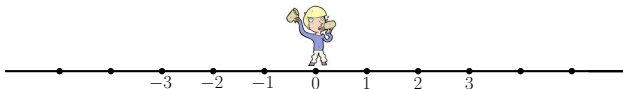
Will he ever get back home (*return to 0*)?

Set p = probability of ever returning to 0.

Since first step is in $\{-1, +1\}$ equally likely, by symmetry

$$p = \mathbb{P}(\text{hit } 0 \text{ eventually starting from } 1).$$

Random walk on \mathbb{Z}



Question

Will he ever get back home (*return to 0*)?

Set p = probability of ever returning to 0.

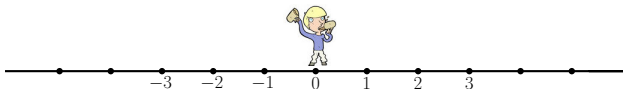
Since first step is in $\{-1, +1\}$ equally likely, by symmetry

$$p = \mathbb{P}(\text{hit 0 eventually starting from 1}).$$

Using symmetry again

$$\mathbb{P}(\text{hit 0 before 2 starting from 1}) = \frac{1}{2}$$

Random walk on \mathbb{Z}



Question

Will he ever get back home (*return to 0*)?

Set p = probability of ever returning to 0.

Since first step is in $\{-1, +1\}$ equally likely, by symmetry

$$p = \mathbb{P}(\text{hit 0 eventually starting from 1}).$$

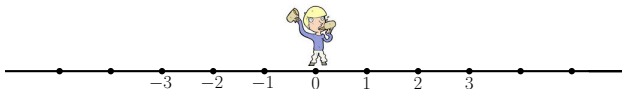
Using symmetry again

$$\mathbb{P}(\text{hit 0 before 2 starting from 1}) = \frac{1}{2}$$

$$\mathbb{P}(\text{hit 0 before 4 starting from 2}) = \frac{1}{2}$$

...

Random walk on \mathbb{Z}



Question

Will he ever get back home (*return to 0*)?

Set p = probability of ever returning to 0.

Since first step is in $\{-1, +1\}$ equally likely, by symmetry

$$p = \mathbb{P}(\text{hit 0 eventually starting from 1}).$$

Using symmetry again

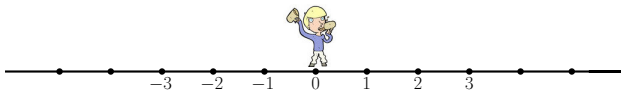
$$\mathbb{P}(\text{hit 0 before 2 starting from 1}) = \frac{1}{2}$$

$$\mathbb{P}(\text{hit 0 before 4 starting from 2}) = \frac{1}{2}$$

...

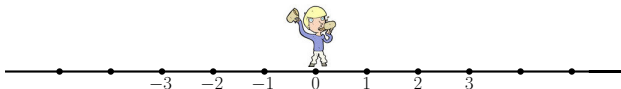
Hence he will eventually hit 0 with probability 1, i.e. $p = 1$.

Random walk on \mathbb{Z}



We showed that with probability 1 the random walk on \mathbb{Z} will return to 0.

Random walk on \mathbb{Z}

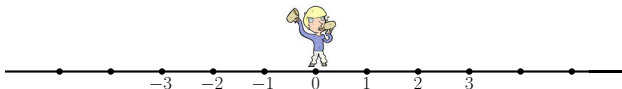


We showed that with probability 1 the random walk on \mathbb{Z} will return to 0.

Question

How long does it take on average to return to 0?

Random walk on \mathbb{Z}



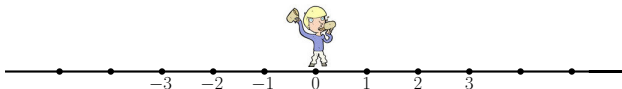
We showed that with probability 1 the random walk on \mathbb{Z} will return to 0.

Question

How long does it take on average to return to 0?

Let's call x the time to hit 0 starting from 1. By translation invariance, once at 2, the average time to hit 0 is $2x$.

Random walk on \mathbb{Z}



We showed that with probability 1 the random walk on \mathbb{Z} will return to 0.

Question

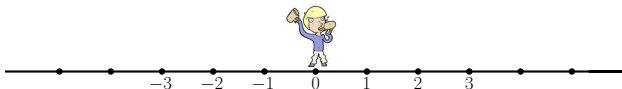
How long does it take on average to return to 0?

Let's call x the time to hit 0 starting from 1. By translation invariance, once at 2, the average time to hit 0 is $2x$.

$$x = 1 + \frac{1}{2} \cdot (2x)$$

Diagram illustrating the equation $x = 1 + \frac{1}{2} \cdot (2x)$. The term 1 is labeled "first step" with an arrow pointing to it. The term $\frac{1}{2} \cdot (2x)$ is labeled "remaining time" with an arrow pointing to it.

Random walk on \mathbb{Z}



We showed that with probability 1 the random walk on \mathbb{Z} will return to 0.

Question

How long does it take on average to return to 0?

Let's call x the time to hit 0 starting from 1. By translation invariance, once at 2, the average time to hit 0 is $2x$.

$$x = 1 + \frac{1}{2} \cdot (2x) \Rightarrow x = \infty$$

first step remaining time

Let's now calculate the probability of being at 0 at time $2n$ when started from 0.

Let's now calculate the probability of being at 0 at time $2n$ when started from 0.

We must make n steps to the right and n steps to the left.

Let's now calculate the probability of being at 0 at time $2n$ when started from 0.

We must make n steps to the right and n steps to the left. There are $\binom{2n}{n}$ ways of doing this.

Let's now calculate the probability of being at 0 at time $2n$ when started from 0.

We must make n steps to the right and n steps to the left. There are $\binom{2n}{n}$ ways of doing this. The probability of every path from 0 to 0 is 2^{-2n} .

Let's now calculate the probability of being at 0 at time $2n$ when started from 0.

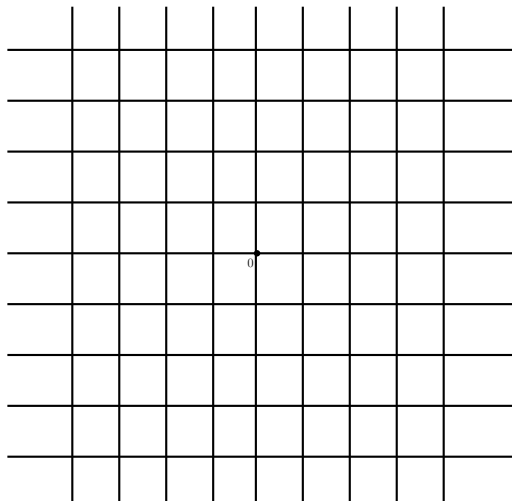
We must make n steps to the right and n steps to the left. There are $\binom{2n}{n}$ ways of doing this. The probability of every path from 0 to 0 is 2^{-2n} .

Therefore

$$\mathbb{P}(\text{hit 0 at time } 2n \text{ starting from 0}) = \binom{2n}{n} \cdot \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}$$

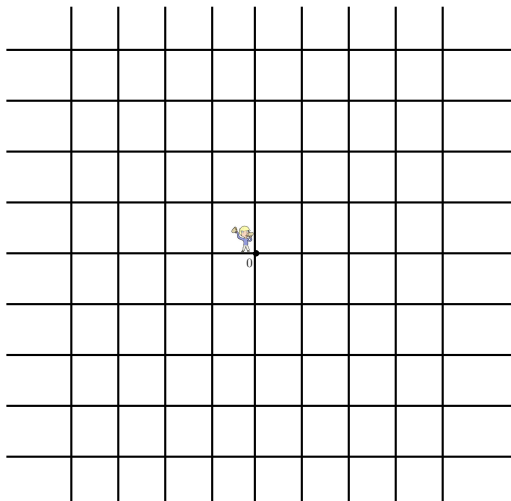
by Stirling's formula.

Random walk in \mathbb{Z}^2



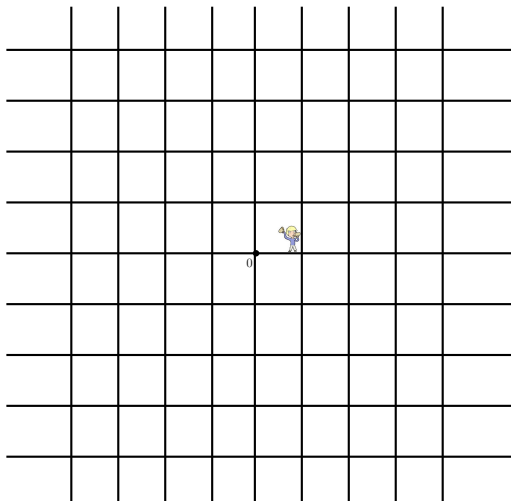
At every time step he chooses one of the four neighbours at random and jumps there. This process is called a **simple symmetric random walk** on \mathbb{Z}^2 .

Random walk in \mathbb{Z}^2



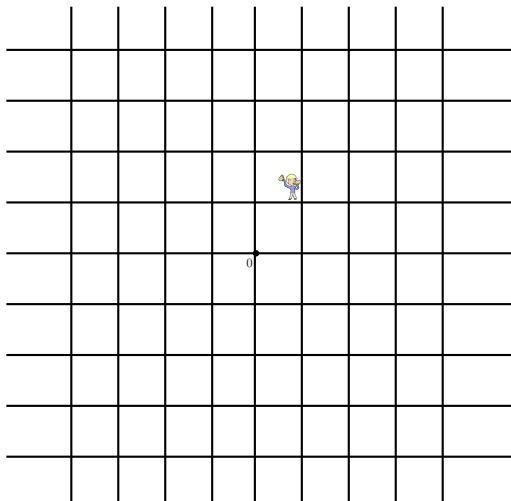
At every time step he chooses one of the four neighbours at random and jumps there. This process is called a **simple symmetric random walk** on \mathbb{Z}^2 .

Random walk in \mathbb{Z}^2



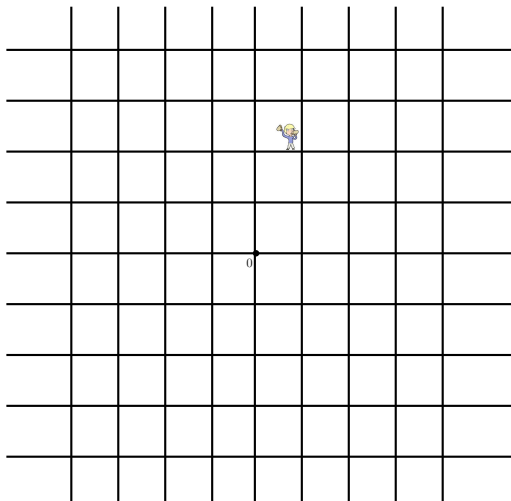
At every time step he chooses one of the four neighbours at random and jumps there. This process is called a **simple symmetric random walk** on \mathbb{Z}^2 .

Random walk in \mathbb{Z}^2



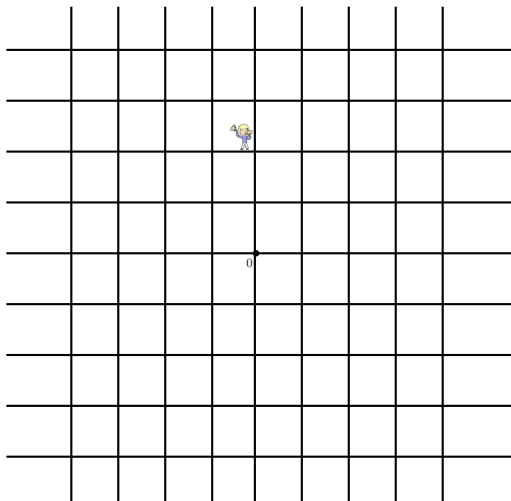
At every time step he chooses one of the four neighbours at random and jumps there. This process is called a **simple symmetric random walk** on \mathbb{Z}^2 .

Random walk in \mathbb{Z}^2



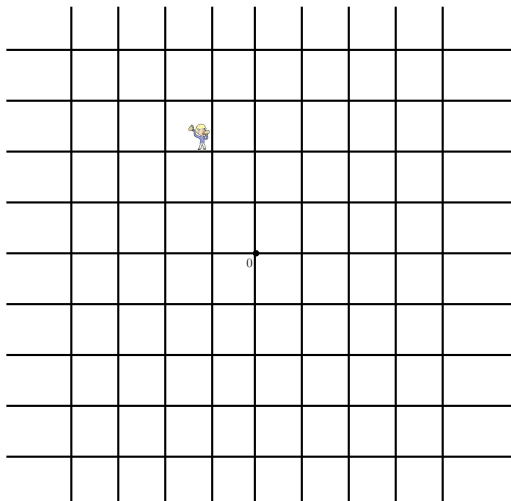
At every time step he chooses one of the four neighbours at random and jumps there. This process is called a **simple symmetric random walk** on \mathbb{Z}^2 .

Random walk in \mathbb{Z}^2



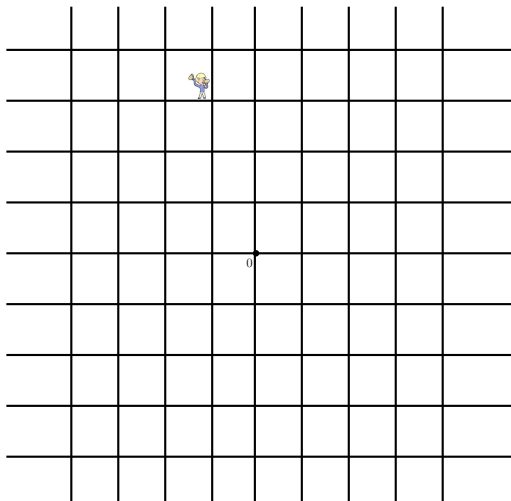
At every time step he chooses one of the four neighbours at random and jumps there. This process is called a **simple symmetric random walk** on \mathbb{Z}^2 .

Random walk in \mathbb{Z}^2



At every time step he chooses one of the four neighbours at random and jumps there. This process is called a **simple symmetric random walk** on \mathbb{Z}^2 .

Random walk in \mathbb{Z}^2



At every time step he chooses one of the four neighbours at random and jumps there. This process is called a **simple symmetric random walk** on \mathbb{Z}^2 .

Question

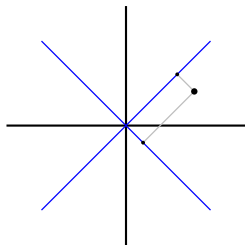
Will he ever get back home (*return to 0*)?

Random walk in \mathbb{Z}^2

Question

Will he ever get back home (*return to 0*)?

First **project** every point in \mathbb{Z}^2 on the diagonal axes, $y = \pm x$.

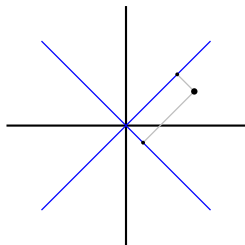


Random walk in \mathbb{Z}^2

Question

Will he ever get back home (*return to 0*)?

First **project** every point in \mathbb{Z}^2 on the diagonal axes, $y = \pm x$.



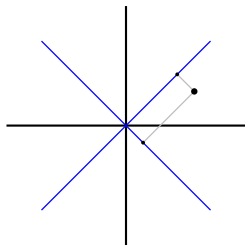
- Get a random walk on $\frac{\mathbb{Z}}{\sqrt{2}} \times \frac{\mathbb{Z}}{\sqrt{2}}$ rotated by 45° .

Random walk in \mathbb{Z}^2

Question

Will he ever get back home (*return to 0*)?

First **project** every point in \mathbb{Z}^2 on the diagonal axes, $y = \pm x$.



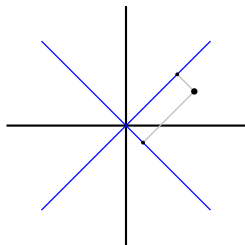
- Get a random walk on $\frac{\mathbb{Z}}{\sqrt{2}} \times \frac{\mathbb{Z}}{\sqrt{2}}$ rotated by 45° .
- When walk moves one step in \mathbb{Z}^2 , each of the two coordinates on the rotated axes moves by $\frac{1}{\sqrt{2}}$.

Random walk in \mathbb{Z}^2

Question

Will he ever get back home (*return to 0*)?

First **project** every point in \mathbb{Z}^2 on the diagonal axes, $y = \pm x$.



- Get a random walk on $\frac{\mathbb{Z}}{\sqrt{2}} \times \frac{\mathbb{Z}}{\sqrt{2}}$ rotated by 45° .
- When walk moves one step in \mathbb{Z}^2 , each of the two coordinates on the rotated axes moves by $\frac{1}{\sqrt{2}}$.
- \implies get two independent SRW's in $\frac{\mathbb{Z}}{\sqrt{2}}$.

Random walk in \mathbb{Z}^2

Question

Will he ever get back home (*return to 0*)?

Question

Will he ever get back home (*return to 0*)?

Let V be the total number of visits to 0.

Question

Will he ever get back home (*return to 0*)?

Let V be the total number of visits to 0. Set $p = \mathbb{P}(\text{return to 0})$.

Question

Will he ever get back home (*return to 0*)?

Let V be the total number of visits to 0. Set $p = \mathbb{P}(\text{return to 0})$. By the Markov property

$$\mathbb{P}(V > r) = p^r.$$

Question

Will he ever get back home (*return to 0*)?

Let V be the total number of visits to 0. Set $p = \mathbb{P}(\text{return to 0})$. By the Markov property

$$\mathbb{P}(V > r) = p^r.$$

Recall the useful formula

$$\mathbb{E}[V] = \sum_{r=0}^{\infty} \mathbb{P}(V > r).$$

Question

Will he ever get back home (*return to 0*)?

Let V be the total number of visits to 0. Set $p = \mathbb{P}(\text{return to 0})$. By the Markov property

$$\mathbb{P}(V > r) = p^r.$$

Recall the useful formula

$$\mathbb{E}[V] = \sum_{r=0}^{\infty} \mathbb{P}(V > r).$$

(*Proof. Write $V = \sum_{r=0}^{\infty} \mathbf{1}(V > r)$ and use linearity of expectation.*)

Question

Will he ever get back home (*return to 0*)?

Let V be the total number of visits to 0. Set $p = \mathbb{P}(\text{return to 0})$. By the Markov property

$$\mathbb{P}(V > r) = p^r.$$

Recall the useful formula

$$\mathbb{E}[V] = \sum_{r=0}^{\infty} \mathbb{P}(V > r).$$

(*Proof. Write $V = \sum_{r=0}^{\infty} \mathbf{1}(V > r)$ and use linearity of expectation.*) If $p \neq 1$, then $\mathbb{E}[V] = \frac{1}{1-p} < \infty$.

Question

Will he ever get back home (*return to 0*)?

Let V be the total number of visits to 0. Set $p = \mathbb{P}(\text{return to } 0)$. By the Markov property

$$\mathbb{P}(V > r) = p^r.$$

Recall the useful formula

$$\mathbb{E}[V] = \sum_{r=0}^{\infty} \mathbb{P}(V > r).$$

(*Proof. Write $V = \sum_{r=0}^{\infty} \mathbf{1}(V > r)$ and use linearity of expectation.*) If $p \neq 1$, then $\mathbb{E}[V] = \frac{1}{1-p} < \infty$. We can also calculate $\mathbb{E}[V]$ using

$$\mathbb{E}[V] = \sum_n \mathbb{P}(\text{return to } 0 \text{ in } 2n \text{ steps starting from } 0).$$

Random walk in \mathbb{Z}^2

Question

Will he ever get back home (*return to 0*)?

Random walk in \mathbb{Z}^2

Question

Will he ever get back home (*return to 0*)?

The random walk will be back at $(0, 0)$ only if the two **projections** are at 0.

Question

Will he ever get back home (*return to 0*)?

The random walk will be back at $(0,0)$ only if the two **projections** are at 0. By the independence of the two projected walks we get

Question

Will he ever get back home (*return to 0*)?

The random walk will be back at $(0, 0)$ only if the two **projections** are at 0. By the independence of the two projected walks we get

$$\mathbb{P}(\text{return to 0 in } 2n \text{ steps starting from 0}) \sim \frac{1}{\pi n}.$$

Question

Will he ever get back home (*return to 0*)?

The random walk will be back at $(0, 0)$ only if the two **projections** are at 0. By the independence of the two projected walks we get

$$\mathbb{P}(\text{return to 0 in } 2n \text{ steps starting from 0}) \sim \frac{1}{\pi n}.$$

Therefore,

$$\mathbb{E}[V] \sim \sum_n \frac{1}{\pi n} = \infty.$$

Question

Will he ever get back home (*return to 0*)?

The random walk will be back at $(0, 0)$ only if the two **projections** are at 0. By the independence of the two projected walks we get

$$\mathbb{P}(\text{return to 0 in } 2n \text{ steps starting from 0}) \sim \frac{1}{\pi n}.$$

Therefore,

$$\mathbb{E}[V] \sim \sum_n \frac{1}{\pi n} = \infty.$$

Recall $p = \mathbb{P}(\text{return to 0})$ and $\mathbb{E}[V] < \infty$ if and only if $p < 1$.

Question

Will he ever get back home (*return to 0*)?

The random walk will be back at $(0, 0)$ only if the two **projections** are at 0. By the independence of the two projected walks we get

$$\mathbb{P}(\text{return to 0 in } 2n \text{ steps starting from 0}) \sim \frac{1}{\pi n}.$$

Therefore,

$$\mathbb{E}[V] \sim \sum_n \frac{1}{\pi n} = \infty.$$

Recall $p = \mathbb{P}(\text{return to 0})$ and $\mathbb{E}[V] < \infty$ if and only if $p < 1$. Therefore $p = 1$.

Question

Will he ever get back home (*return to 0*)?

The random walk will be back at $(0, 0)$ only if the two **projections** are at 0. By the independence of the two projected walks we get

$$\mathbb{P}(\text{return to 0 in } 2n \text{ steps starting from 0}) \sim \frac{1}{\pi n}.$$

Therefore,

$$\mathbb{E}[V] \sim \sum_n \frac{1}{\pi n} = \infty.$$

Recall $p = \mathbb{P}(\text{return to 0})$ and $\mathbb{E}[V] < \infty$ if and only if $p < 1$. Therefore $p = 1$. So with probability 1 the random walk in \mathbb{Z}^2 will return to 0.

Question

How long does it take on average to return to 0?

Question

How long does it take on average to return to 0?

The random walk will be back at $(0,0)$ only if the two **projections** are at 0.

Question

How long does it take on average to return to 0?

The random walk will be back at $(0,0)$ only if the two **projections** are at 0.

Thus it will take ∞ time on average to return.

Random walk in \mathbb{Z}^3





- Let's add one more dimension.

Random walk in \mathbb{Z}^3



- Let's add one more dimension.
- Now at every step the walk chooses one of his **6 neighbours** at random and jumps there.

Random walk in \mathbb{Z}^3



- Let's add one more dimension.
- Now at every step the walk chooses one of his **6 neighbours** at random and jumps there.

Question

Will he ever return to 0 in \mathbb{Z}^3 ?

Random walk in \mathbb{Z}^3



- Let's add one more dimension.
- Now at every step the walk chooses one of his **6 neighbours** at random and jumps there.

Question

Will he ever return to 0 in \mathbb{Z}^3 ?

Unfortunately, here we cannot project as in \mathbb{Z}^2 and obtain **three independent** random walks in \mathbb{Z} .

Random walk in \mathbb{Z}^3



- Let's add one more dimension.
- Now at every step the walk chooses one of his **6 neighbours** at random and jumps there.

Question

Will he ever return to 0 in \mathbb{Z}^3 ?

Unfortunately, here we cannot project as in \mathbb{Z}^2 and obtain **three independent** random walks in \mathbb{Z} . We can calculate the return probability to 0 by summing over all paths and get

$$\mathbb{P}(\text{return to 0 after } 2n \text{ steps}) \asymp \frac{1}{n^{3/2}}.$$

Random walk in \mathbb{Z}^3



- Let's add one more dimension.
- Now at every step the walk chooses one of his **6 neighbours** at random and jumps there.

Question

Will he ever return to 0 in \mathbb{Z}^3 ?

Unfortunately, here we cannot project as in \mathbb{Z}^2 and obtain **three independent** random walks in \mathbb{Z} . We can calculate the return probability to 0 by summing over all paths and get

$$\mathbb{P}(\text{return to 0 after } 2n \text{ steps}) \asymp \frac{1}{n^{3/2}}.$$

Therefore,

$$\sum_n \mathbb{P}(\text{return to 0 after } 2n \text{ steps}) < \infty,$$



- Let's add one more dimension.
- Now at every step the walk chooses one of his **6 neighbours** at random and jumps there.

Question

Will he ever return to 0 in \mathbb{Z}^3 ?

Unfortunately, here we cannot project as in \mathbb{Z}^2 and obtain **three independent** random walks in \mathbb{Z} . We can calculate the return probability to 0 by summing over all paths and get

$$\mathbb{P}(\text{return to 0 after } 2n \text{ steps}) \asymp \frac{1}{n^{3/2}}.$$

Therefore,

$$\sum_n \mathbb{P}(\text{return to 0 after } 2n \text{ steps}) < \infty,$$

meaning that

$$\mathbb{P}(\text{return to 0 at some point}) < 1.$$

Pólya's theorem

Suppose $G = (V, E)$ is a connected graph. We call it **recurrent** if a random walk on it visits its starting location **infinitely many times** and **transient** otherwise.

Pólya's theorem

Suppose $G = (V, E)$ is a connected graph. We call it **recurrent** if a random walk on it visits its starting location **infinitely many times** and **transient** otherwise. From the previous discussion it follows

Pólya's theorem

Suppose $G = (V, E)$ is a connected graph. We call it **recurrent** if a random walk on it visits its starting location **infinitely many times** and **transient** otherwise. From the previous discussion it follows

Theorem (Pólya 1920)

Simple symmetric random walk is recurrent on \mathbb{Z} and \mathbb{Z}^2 and transient on \mathbb{Z}^d for $d \geq 3$.

Pólya's theorem

Suppose $G = (V, E)$ is a connected graph. We call it **recurrent** if a random walk on it visits its starting location **infinitely many times** and **transient** otherwise. From the previous discussion it follows

Theorem (Pólya 1920)

Simple symmetric random walk is recurrent on \mathbb{Z} and \mathbb{Z}^2 and transient on \mathbb{Z}^d for $d \geq 3$.

Let $\bar{G} \subseteq G$ be two graphs.

Pólya's theorem

Suppose $G = (V, E)$ is a connected graph. We call it **recurrent** if a random walk on it visits its starting location **infinitely many times** and **transient** otherwise. From the previous discussion it follows

Theorem (Pólya 1920)

Simple symmetric random walk is recurrent on \mathbb{Z} and \mathbb{Z}^2 and transient on \mathbb{Z}^d for $d \geq 3$.

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?

Pólya's theorem

Suppose $G = (V, E)$ is a connected graph. We call it **recurrent** if a random walk on it visits its starting location **infinitely many times** and **transient** otherwise. From the previous discussion it follows

Theorem (Pólya 1920)

Simple symmetric random walk is recurrent on \mathbb{Z} and \mathbb{Z}^2 and transient on \mathbb{Z}^d for $d \geq 3$.

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?
- If G is transient, is \bar{G} also transient?

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?
- If G is transient, is \bar{G} also transient?

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?
- If G is transient, is \bar{G} also transient?

\mathbb{Z}^3 is transient, but \mathbb{Z} is recurrent.

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?
- If G is transient, is \bar{G} also transient?

\mathbb{Z}^3 is transient, but \mathbb{Z} is recurrent. So let's rephrase the question

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?
- If G is transient, is \bar{G} also transient?

\mathbb{Z}^3 is transient, but \mathbb{Z} is recurrent. So let's rephrase the question

Question

- If G is recurrent, is \bar{G} also recurrent?

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?
- If G is transient, is \bar{G} also transient?

\mathbb{Z}^3 is transient, but \mathbb{Z} is recurrent. So let's rephrase the question

Question

- If G is recurrent, is \bar{G} also recurrent?
- If \bar{G} is transient, is G also transient?

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?
- If G is transient, is \bar{G} also transient?

\mathbb{Z}^3 is transient, but \mathbb{Z} is recurrent. So let's rephrase the question

Question

- If G is recurrent, is \bar{G} also recurrent?
- If \bar{G} is transient, is G also transient?

To answer, we will first explore a beautiful connection between random walks and electrical networks.

Electrical networks

Suppose we have a finite connected graph $G = (V, E)$.

Electrical networks

Suppose we have a finite connected graph $G = (V, E)$. To each edge e there is associated a conductance w_e which is a positive real number.

Electrical networks

Suppose we have a finite connected graph $G = (V, E)$. To each edge e there is associated a conductance w_e which is a positive real number.

The resistance R_e is the inverse of the conductance.

Electrical networks

Suppose we have a finite connected graph $G = (V, E)$. To each edge e there is associated a conductance w_e which is a positive real number.

The resistance R_e is the inverse of the conductance.

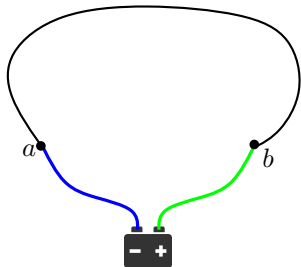
Let a, b be two distinguished vertices of the graph.

Electrical networks

Suppose we have a finite connected graph $G = (V, E)$. To each edge e there is associated a conductance w_e which is a positive real number.

The resistance R_e is the inverse of the conductance.

Let a, b be two distinguished vertices of the graph.

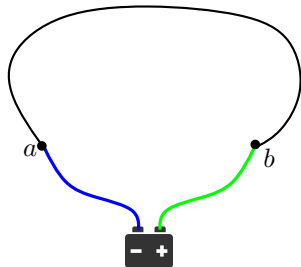


Electrical networks

Suppose we have a finite connected graph $G = (V, E)$. To each edge e there is associated a conductance w_e which is a positive real number.

The resistance R_e is the inverse of the conductance.

Let a, b be two distinguished vertices of the graph.



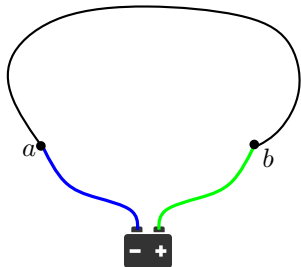
- Suppose we connect a 1-volt battery to a and b .

Electrical networks

Suppose we have a finite connected graph $G = (V, E)$. To each edge e there is associated a conductance w_e which is a positive real number.

The resistance R_e is the inverse of the conductance.

Let a, b be two distinguished vertices of the graph.



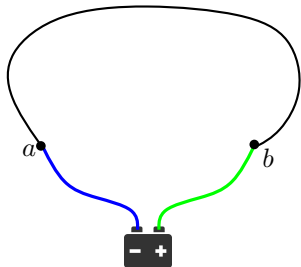
- Suppose we connect a 1-volt battery to a and b .
- This applies **voltage** $v = 1$ at a and $v = 0$ at b .

Electrical networks

Suppose we have a finite connected graph $G = (V, E)$. To each edge e there is associated a conductance w_e which is a positive real number.

The resistance R_e is the inverse of the conductance.

Let a, b be two distinguished vertices of the graph.



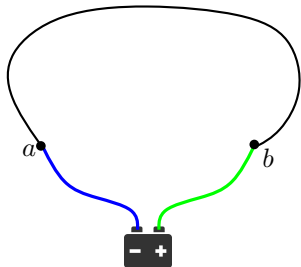
- Suppose we connect a 1-volt battery to a and b .
- This applies **voltage** $v = 1$ at a and $v = 0$ at b .
- This now induces a **current** i that flows from a to b .

Electrical networks

Suppose we have a finite connected graph $G = (V, E)$. To each edge e there is associated a conductance w_e which is a positive real number.

The resistance R_e is the inverse of the conductance.

Let a, b be two distinguished vertices of the graph.



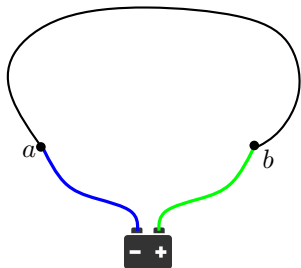
- Suppose we connect a 1-volt battery to a and b .
- This applies **voltage** $v = 1$ at a and $v = 0$ at b .
- This now induces a **current** i that flows from a to b . Along every edge $e = \langle x, y \rangle$ a current i_{xy} ($i_{xy} = -i_{yx}$) will flow and there will be a potential difference $v(x) - v(y)$.

Electrical networks

Suppose we have a finite connected graph $G = (V, E)$. To each edge e there is associated a conductance w_e which is a positive real number.

The resistance R_e is the inverse of the conductance.

Let a, b be two distinguished vertices of the graph.



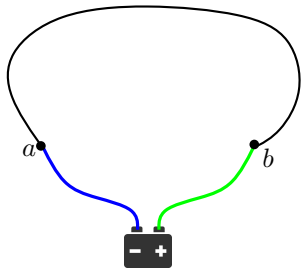
- Suppose we connect a 1-volt battery to a and b .
- This applies **voltage** $v = 1$ at a and $v = 0$ at b .
- This now induces a **current** i that flows from a to b . Along every edge $e = \langle x, y \rangle$ a current i_{xy} ($i_{xy} = -i_{yx}$) will flow and there will be a potential difference $v(x) - v(y)$.
- **Kirchoff's laws:**

Electrical networks

Suppose we have a finite connected graph $G = (V, E)$. To each edge e there is associated a conductance w_e which is a positive real number.

The resistance R_e is the inverse of the conductance.

Let a, b be two distinguished vertices of the graph.



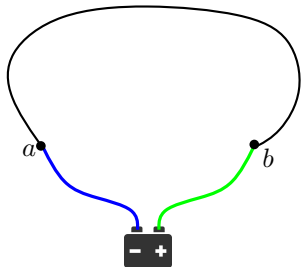
- Suppose we connect a 1-volt battery to a and b .
- This applies **voltage** $v = 1$ at a and $v = 0$ at b .
- This now induces a **current** i that flows from a to b . Along every edge $e = \langle x, y \rangle$ a current i_{xy} ($i_{xy} = -i_{yx}$) will flow and there will be a potential difference $v(x) - v(y)$.
- **Kirchoff's laws:**
 - sum of voltage differences along closed circuits = 0

Electrical networks

Suppose we have a finite connected graph $G = (V, E)$. To each edge e there is associated a conductance w_e which is a positive real number.

The resistance R_e is the inverse of the conductance.

Let a, b be two distinguished vertices of the graph.



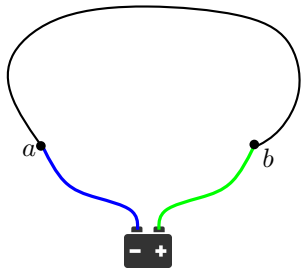
- Suppose we connect a 1-volt battery to a and b .
- This applies **voltage** $v = 1$ at a and $v = 0$ at b .
- This now induces a **current** i that flows from a to b . Along every edge $e = \langle x, y \rangle$ a current i_{xy} ($i_{xy} = -i_{yx}$) will flow and there will be a potential difference $v(x) - v(y)$.
- **Kirchoff's laws:**
 - sum of voltage differences along closed circuits = 0
 - current into x = current out of x

Electrical networks

Suppose we have a finite connected graph $G = (V, E)$. To each edge e there is associated a conductance w_e which is a positive real number.

The resistance R_e is the inverse of the conductance.

Let a, b be two distinguished vertices of the graph.



- Suppose we connect a 1-volt battery to a and b .
- This applies **voltage** $v = 1$ at a and $v = 0$ at b .
- This now induces a **current** i that flows from a to b . Along every edge $e = \langle x, y \rangle$ a current i_{xy} ($i_{xy} = -i_{yx}$) will flow and there will be a potential difference $v(x) - v(y)$.
- **Kirchoff's laws:**
 - sum of voltage differences along closed circuits = 0
 - current into x = current out of x
- **Ohm's law:** $v(x) - v(y) = i_{xy}R_{xy}$

- **Energy** dissipated along the edge $\langle x, y \rangle$ when current flows is $E_{xy} = i_{xy}(v(x) - v(y))$.

- **Energy** dissipated along the edge $\langle x, y \rangle$ when current flows is $E_{xy} = i_{xy}(v(x) - v(y))$.
- By **Ohm's law** $E_{xy} = i_{xy}^2 R_{xy}$.

- **Energy** dissipated along the edge $\langle x, y \rangle$ when current flows is $E_{xy} = i_{xy}(v(x) - v(y))$.
- By **Ohm's law** $E_{xy} = i_{xy}^2 R_{xy}$.
- As in any physical system, energy is an additive quantity

$$E = \sum_{e=\langle x,y \rangle} i_{xy}^2 R_{xy}$$

- **Energy** dissipated along the edge $\langle x, y \rangle$ when current flows is $E_{xy} = i_{xy}(v(x) - v(y))$.
- By **Ohm's law** $E_{xy} = i_{xy}^2 R_{xy}$.
- As in any physical system, energy is an additive quantity

$$E = \sum_{e=\langle x,y \rangle} i_{xy}^2 R_{xy}$$

- Basic algebra $\implies E = (v(a) - v(b))i_a$, where $i_a = \sum_x i_{ax}$.

- **Energy** dissipated along the edge $\langle x, y \rangle$ when current flows is $E_{xy} = i_{xy}(v(x) - v(y))$.
- By **Ohm's law** $E_{xy} = i_{xy}^2 R_{xy}$.
- As in any physical system, energy is an additive quantity

$$E = \sum_{e=\langle x,y \rangle} i_{xy}^2 R_{xy}$$

- Basic algebra $\implies E = (v(a) - v(b))i_a$, where $i_a = \sum_x i_{ax}$.
- Define $R_{\text{eff}}(a, b) = \frac{v(a) - v(b)}{i_a}$, the **effective resistance at a, b** .

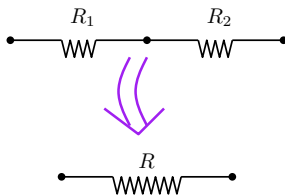
- **Energy** dissipated along the edge $\langle x, y \rangle$ when current flows is $E_{xy} = i_{xy}(v(x) - v(y))$.
- By **Ohm's law** $E_{xy} = i_{xy}^2 R_{xy}$.
- As in any physical system, energy is an additive quantity

$$E = \sum_{e=\langle x,y \rangle} i_{xy}^2 R_{xy}$$

- Basic algebra $\implies E = (v(a) - v(b))i_a$, where $i_a = \sum_x i_{ax}$.
- Define $R_{\text{eff}}(a, b) = \frac{v(a) - v(b)}{i_a}$, the **effective resistance at a, b** .
- If we ignore the network and imagine only one resistance R_{eff} between a and b , then same current will flow and same energy dissipated.

Rules for calculating effective resistance

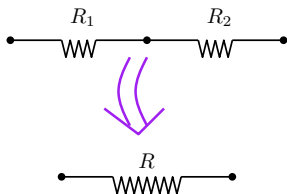
- **Series rule:** If two resistors R_1 and R_2 are connected in series, it is equivalent to a single resistor



$$R = R_1 + R_2$$

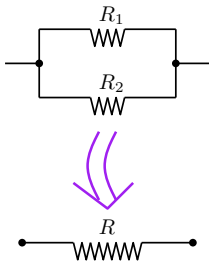
Rules for calculating effective resistance

- **Series rule:** If two resistors R_1 and R_2 are connected in series, it is equivalent to a single resistor



$$R = R_1 + R_2$$

- **Parallel law:** If they are connected in parallel, then



$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Definition

A flow from a to b is an assignment of numbers $\{j_{xy}\}$ on the edges of the graph such that

- $j_{xy} = -j_{yx}$
- $j_{xy} = 0$ if $\langle x, y \rangle$ is not an edge
- $\sum_y j_{xy} = 0$ if $x \neq a, b$.

Definition

A flow from a to b is an assignment of numbers $\{j_{xy}\}$ on the edges of the graph such that

- $j_{xy} = -j_{yx}$
- $j_{xy} = 0$ if $\langle x, y \rangle$ is not an edge
- $\sum_y j_{xy} = 0$ if $x \neq a, b$.



Thomson's Variational Principle Unit flows satisfying Kirchoff's laws minimize energy dissipation among all unit flows from a to b .

Definition

A flow from a to b is an assignment of numbers $\{j_{xy}\}$ on the edges of the graph such that

- $j_{xy} = -j_{yx}$
- $j_{xy} = 0$ if $\langle x, y \rangle$ is not an edge
- $\sum_y j_{xy} = 0$ if $x \neq a, b$.



Thomson's Variational Principle Unit flows satisfying Kirchoff's laws minimize energy dissipation among all unit flows from a to b .

(William Thomson, 1st Baron Kelvin 1824 – 1907)

Definition

A flow from a to b is an assignment of numbers $\{j_{xy}\}$ on the edges of the graph such that

- $j_{xy} = -j_{yx}$
- $j_{xy} = 0$ if $\langle x, y \rangle$ is not an edge
- $\sum_y j_{xy} = 0$ if $x \neq a, b$.



Thomson's Variational Principle Unit flows satisfying Kirchoff's laws minimize energy dissipation among all unit flows from a to b .

(William Thomson, 1st Baron Kelvin 1824 – 1907)



Rayleigh's monotonicity principle R_{eff} is non-decreasing as a function of the component resistances.

Definition

A flow from a to b is an assignment of numbers $\{j_{xy}\}$ on the edges of the graph such that

- $j_{xy} = -j_{yx}$
- $j_{xy} = 0$ if $\langle x, y \rangle$ is not an edge
- $\sum_y j_{xy} = 0$ if $x \neq a, b$.



Thomson's Variational Principle Unit flows satisfying Kirchoff's laws minimize energy dissipation among all unit flows from a to b .

(William Thomson, 1st Baron Kelvin 1824 – 1907)



Rayleigh's monotonicity principle R_{eff} is non-decreasing as a function of the component resistances.

(John William Strutt, 3rd Baron Rayleigh 1842 – 1919)

Electrical networks and random walks

Suppose we have a graph $G = (V, E)$ with weights (w_{xy}) on the edges.

Suppose we have a graph $G = (V, E)$ with weights (w_{xy}) on the edges.

Definition

A function $f : V \rightarrow \mathbb{R}$ is called **harmonic** on $W \subseteq V$ if

$$f(i) = \sum_j \frac{w_{ij}}{\sum_k w_{ik}} \cdot f(j) \quad \text{for all } i \in W.$$

Suppose we have a graph $G = (V, E)$ with weights (w_{xy}) on the edges.

Definition

A function $f : V \rightarrow \mathbb{R}$ is called **harmonic** on $W \subseteq V$ if

$$f(i) = \sum_j \frac{w_{ij}}{\sum_k w_{ik}} \cdot f(j) \quad \text{for all } i \in W.$$

Uniqueness principle Two harmonic functions defined on the same space with the same boundary conditions must be equal.

Electrical networks and random walks

A random walk on G moves by choosing every time a neighbour with probability $P(x, y) = \frac{w_{xy}}{\sum_z w_{xz}}$, i.e. proportional to the weight of the edge connecting them.

Electrical networks and random walks

A random walk on G moves by choosing every time a neighbour with probability $P(x, y) = \frac{w_{xy}}{\sum_z w_{xz}}$, i.e. proportional to the weight of the edge connecting them.

For $x \in V$ let

$$h(x) = \mathbb{P}(\text{reach } a \text{ before } b \text{ starting from } x).$$

A random walk on G moves by choosing every time a neighbour with probability $P(x, y) = \frac{w_{xy}}{\sum_z w_{xz}}$, i.e. proportional to the weight of the edge connecting them.

For $x \in V$ let

$$h(x) = \mathbb{P}(\text{reach } a \text{ before } b \text{ starting from } x).$$

Condition on the first step to get

$$h(a) = 1, h(b) = 0 \quad \text{and} \quad h(x) = \sum_y P(x, y)h(y).$$

A random walk on G moves by choosing every time a neighbour with probability $P(x, y) = \frac{w_{xy}}{\sum_z w_{xz}}$, i.e. proportional to the weight of the edge connecting them.

For $x \in V$ let

$$h(x) = \mathbb{P}(\text{reach } a \text{ before } b \text{ starting from } x).$$

Condition on the first step to get

$$h(a) = 1, h(b) = 0 \quad \text{and} \quad h(x) = \sum_y P(x, y)h(y).$$

$\implies h$ is a harmonic function.

A random walk on G moves by choosing every time a neighbour with probability $P(x, y) = \frac{w_{xy}}{\sum_z w_{xz}}$, i.e. proportional to the weight of the edge connecting them.

For $x \in V$ let

$$h(x) = \mathbb{P}(\text{reach } a \text{ before } b \text{ starting from } x).$$

Condition on the first step to get

$$h(a) = 1, h(b) = 0 \quad \text{and} \quad h(x) = \sum_y P(x, y)h(y).$$

$\implies h$ is a harmonic function.

Regard now the graph as an electrical network with resistances $R_{xy} = w_{xy}^{-1}$. Then the voltage v is a harmonic function and $v(a) = 1$ and $v(b) = 0$.

Electrical networks and random walks

A random walk on G moves by choosing every time a neighbour with probability $P(x, y) = \frac{w_{xy}}{\sum_z w_{xz}}$, i.e. proportional to the weight of the edge connecting them.

For $x \in V$ let

$$h(x) = \mathbb{P}(\text{reach } a \text{ before } b \text{ starting from } x).$$

Condition on the first step to get

$$h(a) = 1, h(b) = 0 \quad \text{and} \quad h(x) = \sum_y P(x, y)h(y).$$

$\implies h$ is a harmonic function.

Regard now the graph as an electrical network with resistances $R_{xy} = w_{xy}^{-1}$. Then the voltage v is a harmonic function and $v(a) = 1$ and $v(b) = 0$.

By the **uniqueness principle** $\boxed{h = v}$.

Electrical networks and random walks

We define the **escape probability**

$$p_{\text{esc}} = \mathbb{P}(\text{reach } b \text{ before returning to } a).$$

Electrical networks and random walks

We define the **escape probability**

$$p_{\text{esc}} = \mathbb{P}(\text{reach } b \text{ before returning to } a).$$

Conditioning on the first step we get

$$1 - p_{\text{esc}} = \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot h(y) = \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot v(y).$$

Electrical networks and random walks

We define the **escape probability**

$$p_{\text{esc}} = \mathbb{P}(\text{reach } b \text{ before returning to } a).$$

Conditioning on the first step we get

$$1 - p_{\text{esc}} = \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot h(y) = \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot v(y).$$

Recall $R_{\text{eff}}(a, b) = \frac{v(a) - v(b)}{i_a} = \frac{1}{i_a}$ (since we applied a 1-volt battery.)

Electrical networks and random walks

We define the **escape probability**

$$p_{\text{esc}} = \mathbb{P}(\text{reach } b \text{ before returning to } a).$$

Conditioning on the first step we get

$$1 - p_{\text{esc}} = \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot h(y) = \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot v(y).$$

Recall $R_{\text{eff}}(a, b) = \frac{v(a) - v(b)}{i_a} = \frac{1}{i_a}$ (since we applied a 1-volt battery.)

$$i_a = \sum_{y \sim a} i_{ay} = \sum_{y \sim a} (v(a) - v(y))w_{ay} = w_a \cdot \left(1 - \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot v(y)\right).$$

Electrical networks and random walks

We define the **escape probability**

$$p_{\text{esc}} = \mathbb{P}(\text{reach } b \text{ before returning to } a).$$

Conditioning on the first step we get

$$1 - p_{\text{esc}} = \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot h(y) = \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot v(y).$$

Recall $R_{\text{eff}}(a, b) = \frac{v(a) - v(b)}{i_a} = \frac{1}{i_a}$ (since we applied a 1-volt battery.)

$$i_a = \sum_{y \sim a} i_{ay} = \sum_{y \sim a} (v(a) - v(y))w_{ay} = w_a \cdot \left(1 - \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot v(y)\right).$$

Therefore

$$p_{\text{esc}} = \frac{i_a}{w_a} = \frac{1}{w_a R_{\text{eff}}(a, b)}$$

Electrical networks and random walks

We define the **escape probability**

$$p_{\text{esc}} = \mathbb{P}(\text{reach } b \text{ before returning to } a).$$

Conditioning on the first step we get

$$1 - p_{\text{esc}} = \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot h(y) = \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot v(y).$$

Recall $R_{\text{eff}}(a, b) = \frac{v(a) - v(b)}{i_a} = \frac{1}{i_a}$ (since we applied a 1-volt battery.)

$$i_a = \sum_{y \sim a} i_{ay} = \sum_{y \sim a} (v(a) - v(y))w_{ay} = w_a \cdot \left(1 - \sum_{y \sim a} \frac{w_{ay}}{w_a} \cdot v(y)\right).$$

Therefore

$$p_{\text{esc}} = \frac{i_a}{w_a} = \frac{1}{w_a R_{\text{eff}}(a, b)}$$

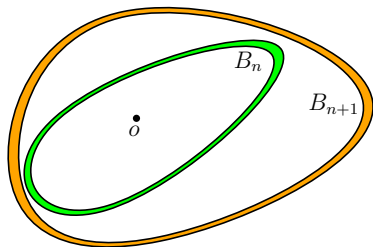
Crucial link between random walks and electrical networks.

Electrical networks and random walks

Let G be an infinite graph and o a distinguished vertex.

Electrical networks and random walks

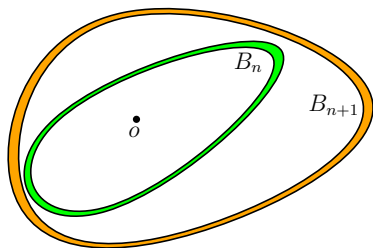
Let G be an infinite graph and o a distinguished vertex.



- Let B_n be the points at distance n from o .

Electrical networks and random walks

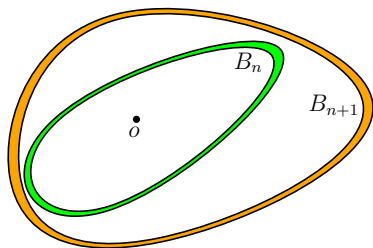
Let G be an infinite graph and o a distinguished vertex.



- Let B_n be the points at distance n from o .
- Glue them to a single point b_n (*Put 0 resistances between them*)

Electrical networks and random walks

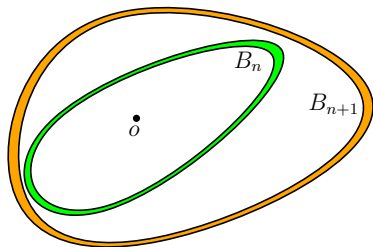
Let G be an infinite graph and o a distinguished vertex.



- Let B_n be the points at distance n from o .
- Glue them to a single point b_n (*Put 0 resistances between them*)
- $\mathbb{P}_o(\text{hit } B_n \text{ before } o) = \frac{1}{w_o R_{\text{eff}}(o, b_n)}$

Electrical networks and random walks

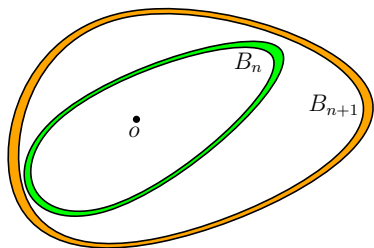
Let G be an infinite graph and o a distinguished vertex.



- Let B_n be the points at distance n from o .
- Glue them to a single point b_n (*Put 0 resistances between them*)
- $\mathbb{P}_o(\text{hit } B_n \text{ before } o) = \frac{1}{w_o R_{\text{eff}}(o, b_n)}$
- Rayleigh monotonicity principle
 $\implies R_{\text{eff}}(o, b_n) \nearrow R_{\text{eff}}(o, \infty)$ as $n \rightarrow \infty$

Electrical networks and random walks

Let G be an infinite graph and o a distinguished vertex.



- Let B_n be the points at distance n from o .
- Glue them to a single point b_n (*Put 0 resistances between them*)
- $\mathbb{P}_o(\text{hit } B_n \text{ before } o) = \frac{1}{w_o R_{\text{eff}}(o, b_n)}$
- Rayleigh monotonicity principle
 $\implies R_{\text{eff}}(o, b_n) \nearrow R_{\text{eff}}(o, \infty)$ as $n \rightarrow \infty$
- $\mathbb{P}(\text{never returns to } o) = \frac{1}{w_o R_{\text{eff}}(o, \infty)}$

$$\text{Recall } \mathbb{P}(\text{never returns to } o) = \frac{1}{w_o R_{\text{eff}}(o, \infty)}$$

$$\text{Recall } \mathbb{P}(\text{never returns to } o) = \frac{1}{w_o R_{\text{eff}}(o, \infty)}$$

- $R_{\text{eff}}(o, \infty) = \infty \iff \mathbb{P}(\text{never returns to } o) = 0 \implies$ walk is **recurrent**.

$$\text{Recall } \mathbb{P}(\text{never returns to } o) = \frac{1}{w_o R_{\text{eff}}(o, \infty)}$$

- $R_{\text{eff}}(o, \infty) = \infty \iff \mathbb{P}(\text{never returns to } o) = 0 \implies$ walk is **recurrent**.
- $R_{\text{eff}}(o, \infty) < \infty \iff \mathbb{P}(\text{never returns to } o) > 0 \implies$ walk is **transient**.

$$\text{Recall } \mathbb{P}(\text{never returns to } o) = \frac{1}{w_o R_{\text{eff}}(o, \infty)}$$

- $R_{\text{eff}}(o, \infty) = \infty \iff \mathbb{P}(\text{never returns to } o) = 0 \implies$ walk is **recurrent**.
- $R_{\text{eff}}(o, \infty) < \infty \iff \mathbb{P}(\text{never returns to } o) > 0 \implies$ walk is **transient**.

Let's go back to the question from earlier :

$$\text{Recall } \mathbb{P}(\text{never returns to } o) = \frac{1}{w_o R_{\text{eff}}(o, \infty)}$$

- $R_{\text{eff}}(o, \infty) = \infty \iff \mathbb{P}(\text{never returns to } o) = 0 \implies$ walk is **recurrent**.
- $R_{\text{eff}}(o, \infty) < \infty \iff \mathbb{P}(\text{never returns to } o) > 0 \implies$ walk is **transient**.

Let's go back to the question from earlier : Suppose $\bar{G} \subseteq G$.

Question

- If G is recurrent, is \bar{G} also recurrent?
- If \bar{G} is transient, is G also transient?

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?
- If \bar{G} is transient, is G also transient?

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?
- If \bar{G} is transient, is G also transient?

We established $\text{walk is recurrent} \iff R_{\text{eff}} = \infty$

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?
- If \bar{G} is transient, is G also transient?

We established walk is recurrent $\iff R_{\text{eff}} = \infty$

\bar{G} is G after replacing finite resistances by ∞ ones.

Transience and recurrence

Let $\bar{G} \subseteq G$ be two graphs.

Question

- If G is recurrent, is \bar{G} also recurrent?
- If \bar{G} is transient, is G also transient?

We established $\text{walk is recurrent} \iff R_{\text{eff}} = \infty$

\bar{G} is G after replacing finite resistances by ∞ ones.

$$R_{\text{eff}}(G) = \infty \implies R_{\text{eff}}(\bar{G}) = \infty$$

Recall monotonicity principle

$$R_{\text{eff}}(\bar{G}) < \infty \implies R_{\text{eff}}(G) < \infty$$

At the hotel there lived also some students with whom I usually took my meals and had friendly relations. On a certain day one of them expected the visit of his fiancée, what I knew [sic], but I did not foresee that he and his fiancée would also set out for a stroll in the woods, and then suddenly I met them there. And then I met them the same morning repeatedly. I don't remember how many times, but certainly much too often and I felt embarrassed: It looked as if I was snooping around which was, I assure you, not the case.

[G. Pólya, Two incidents, in **Scientists at Work**]

Collisions of random walks

At the hotel there lived also some students with whom I usually took my meals and had friendly relations. On a certain day one of them expected the visit of his fiancée, what I knew [sic], but I did not foresee that he and his fiancée would also set out for a stroll in the woods, and then suddenly I met them there. And then I met them the same morning repeatedly. I don't remember how many times, but certainly much too often and I felt embarrassed: It looked as if I was snooping around which was, I assure you, not the case.

[G. Pólya, Two incidents, in Scientists at Work]

Thus, in fact, Pólya's motivation was to understand the number of collisions C of two independent random walks. For random walks on a lattice, this is equivalent to considering returns to zero for a new random walk.

Collisions of random walks

In the lattice \mathbb{Z}^d the question of collisions of two independent walks is equivalent to the question of transience or recurrence of a single walk.

Collisions of random walks

In the lattice \mathbb{Z}^d the question of collisions of two independent walks is equivalent to the question of transience or recurrence of a single walk.

Question

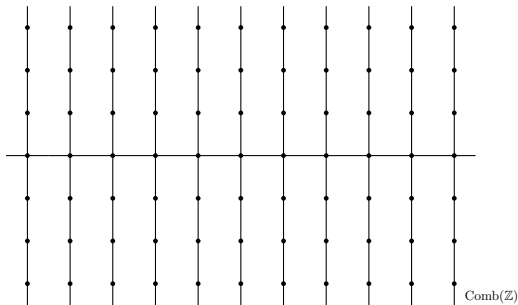
Is this true for general graphs? If a graph is recurrent, will two walks collide infinitely often?

Collisions of random walks

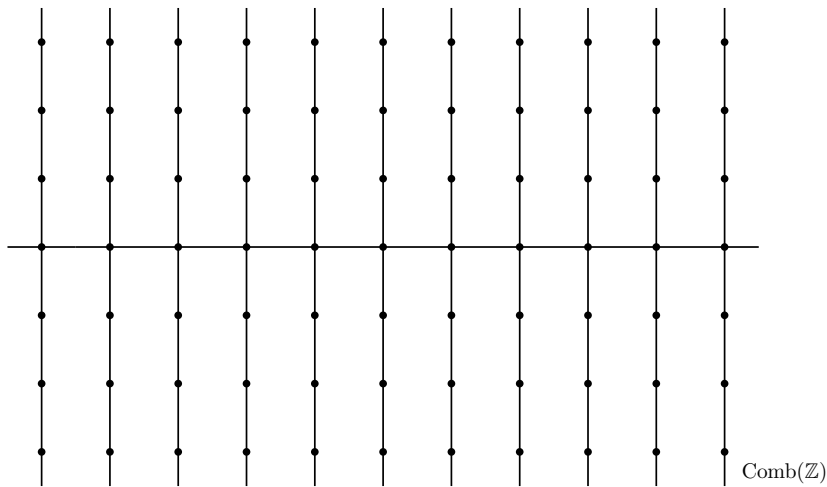
In the lattice \mathbb{Z}^d the question of collisions of two independent walks is equivalent to the question of transience or recurrence of a single walk.

Question

Is this true for general graphs? If a graph is recurrent, will two walks collide infinitely often?



Collisions of random walks



[**Krishnapur – Peres 2004**] Two independent random walks only collide a finite number of times.

Collisions of random walks

The collision property is not well understood.

Collisions of random walks

The collision property is not well understood.

Important: It is not a monotone property!

Collisions of random walks

The collision property is not well understood.

Important: It is not a monotone property!

$$\mathbb{Z} \subseteq \text{Comb}(\mathbb{Z}) \subseteq \mathbb{Z}^2$$

Collisions of random walks

The collision property is not well understood.

Important: It is not a monotone property!

$$\mathbb{Z} \subseteq \text{Comb}(\mathbb{Z}) \subseteq \mathbb{Z}^2$$

Open question

Suppose G has the infinite collision property. If we add one more edge, will it still have the same property?

Self-interacting random walks

So far all the walks satisfied the *Markov property*:

Self-interacting random walks

So far all the walks satisfied the *Markov property*:

past and future independent given the present

Self-interacting random walks

So far all the walks satisfied the *Markov property*:

past and future independent given the present

Suppose that on the first visit to a vertex the random walk behaves differently to later visits to the same vertex.

Self-interacting random walks

So far all the walks satisfied the *Markov property*:

past and future independent given the present

Suppose that on the first visit to a vertex the random walk behaves differently to later visits to the same vertex.

Examples of such walks:

Self-interacting random walks

So far all the walks satisfied the *Markov property*:

past and future independent given the present

Suppose that on the first visit to a vertex the random walk behaves differently to later visits to the same vertex.

Examples of such walks:

- on \mathbb{Z}^2 , on the first visit move horizontally, on later visits move vertically

Self-interacting random walks

So far all the walks satisfied the *Markov property*:

past and future independent given the present

Suppose that on the first visit to a vertex the random walk behaves differently to later visits to the same vertex.

Examples of such walks:

- on \mathbb{Z}^2 , on the first visit move horizontally, on later visits move vertically
- on \mathbb{Z}^3 , on the first visit move in \mathbb{Z} , on later visits move on \mathbb{Z}^2

Self-interacting random walks

So far all the walks satisfied the *Markov property*:

past and future independent given the present

Suppose that on the first visit to a vertex the random walk behaves differently to later visits to the same vertex.

Examples of such walks:

- on \mathbb{Z}^2 , on the first visit move horizontally, on later visits move vertically
- on \mathbb{Z}^3 , on the first visit move in \mathbb{Z} , on later visits move on \mathbb{Z}^2

Question

Which of the above walks are transient and which are recurrent?

Self-interacting random walks

So far all the walks satisfied the *Markov property*:

past and future independent given the present

Suppose that on the first visit to a vertex the random walk behaves differently to later visits to the same vertex.

Examples of such walks:

- on \mathbb{Z}^2 , on the first visit move horizontally, on later visits move vertically
- on \mathbb{Z}^3 , on the first visit move in \mathbb{Z} , on later visits move on \mathbb{Z}^2
Transient [Peres, Schapira, S.]

Question

Which of the above walks are transient and which are recurrent?

Self-interacting random walks

So far all the walks satisfied the *Markov property*:

past and future independent given the present

Suppose that on the first visit to a vertex the random walk behaves differently to later visits to the same vertex.

Examples of such walks:

- on \mathbb{Z}^2 , on the first visit move horizontally, on later visits move vertically **Open question**
- on \mathbb{Z}^3 , on the first visit move in \mathbb{Z} , on later visits move on \mathbb{Z}^2
Transient [Peres, Schapira, S.]

Question

Which of the above walks are transient and which are recurrent?

Edge-reinforced random walks

On \mathbb{Z}^d assign weight 1 to every edge. Every time the walk traverses an edge increase the weight by w . The walk chooses a neighbour to jump to with probability proportional to the weight of the edge connecting them. The resulting walk is called **edge-reinforced random walk**.

Edge-reinforced random walks

On \mathbb{Z}^d assign weight 1 to every edge. Every time the walk traverses an edge increase the weight by w . The walk chooses a neighbour to jump to with probability proportional to the weight of the edge connecting them. The resulting walk is called **edge-reinforced random walk**.

Theorem (Disertori, Sabot and Tarrès 2014)

In $d \geq 3$ if w is small, the resulting walk is transient.

Edge-reinforced random walks

On \mathbb{Z}^d assign weight 1 to every edge. Every time the walk traverses an edge increase the weight by w . The walk chooses a neighbour to jump to with probability proportional to the weight of the edge connecting them. The resulting walk is called **edge-reinforced random walk**.

Theorem (Disertori, Sabot and Tarrès 2014)

In $d \geq 3$ if w is small, the resulting walk is transient.

Theorem (Sabot and Tarrès, Angel, Crawford and Kozma 2012)

For all dimensions if w is sufficiently large, then the resulting walk is recurrent.

Edge-reinforced random walks

On \mathbb{Z}^d assign weight 1 to every edge. Every time the walk traverses an edge increase the weight by w . The walk chooses a neighbour to jump to with probability proportional to the weight of the edge connecting them. The resulting walk is called **edge-reinforced random walk**.

Theorem (Disertori, Sabot and Tarrès 2014)

In $d \geq 3$ if w is small, the resulting walk is transient.

Theorem (Sabot and Tarrès, Angel, Crawford and Kozma 2012)

For all dimensions if w is sufficiently large, then the resulting walk is recurrent.

Open question

For $d = 2$ if w is small, is the resulting walk recurrent?



Thanks!