“Carry on taking fluids!”

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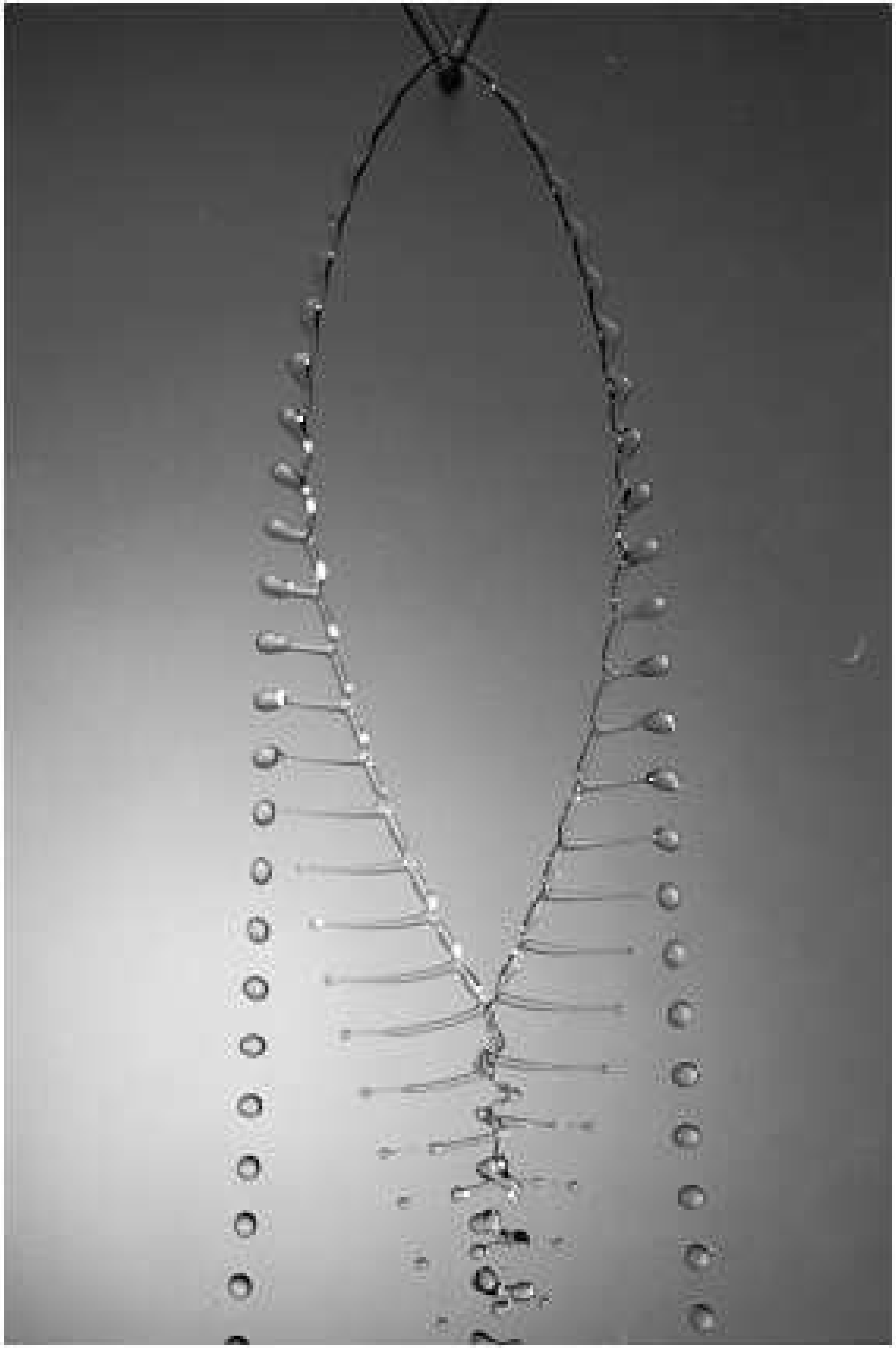
Outline

- **Introduction**
- **My cup runneth over**
- **Flow in a straight pipe or duct**
- **Flow in a slightly curved pipe**
- **Extended Series solutions**
- **Generalized Padé Approximants**
- **Is the zero-R_e flow REALLY unique?**

So what's so great about Fluids?

- They're everywhere!
- They're beautiful.



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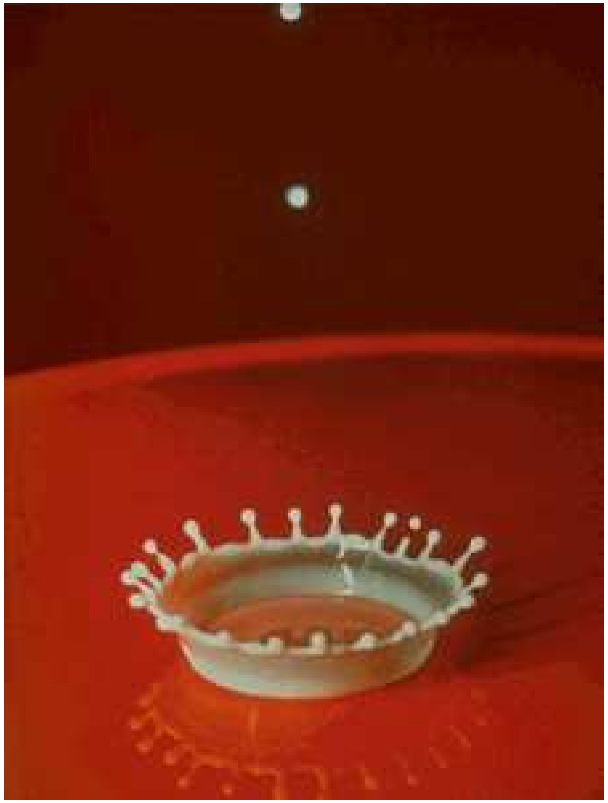

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- They're mysterious. Why do winds flow along isobars on pressure maps? Why do firemen pull on their hoses? Why do bubbles on the top of a cup of tea move to the middle when you stir it? And why do the tea-leaves on the bottom of the cup go towards the middle? When a train enters a tunnel, sometimes the windows blow open, and sometimes shut. Why? And does anyone REALLY believe aeroplanes can fly?
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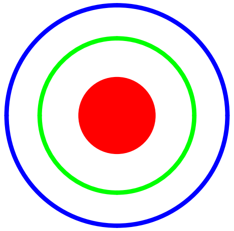

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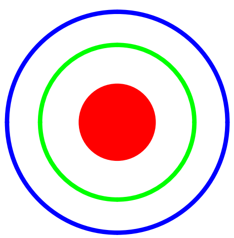

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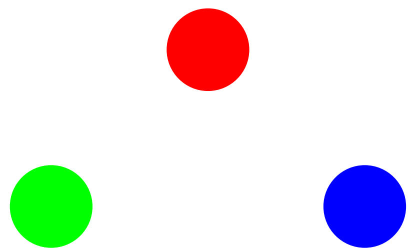

Frequent Reality

The Navier-Stokes equations

The flow of an incompressible Newtonian fluid is governed by the equations

$$\nabla \cdot \mathbf{u} = 0,$$

which ensures that the velocity vector field $\mathbf{u}$ conserves mass, and

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p \, (+\mathbf{F}) + \frac{1}{R_e} \nabla^2 \mathbf{u},$$

which is the momentum balance. The scalar field p is the fluid pressure, which must be found as part of the solution, while $\mathbf{F}$ is any externally imposed force density.
We are interested in steady solutions to these equations, i.e. equilibria.

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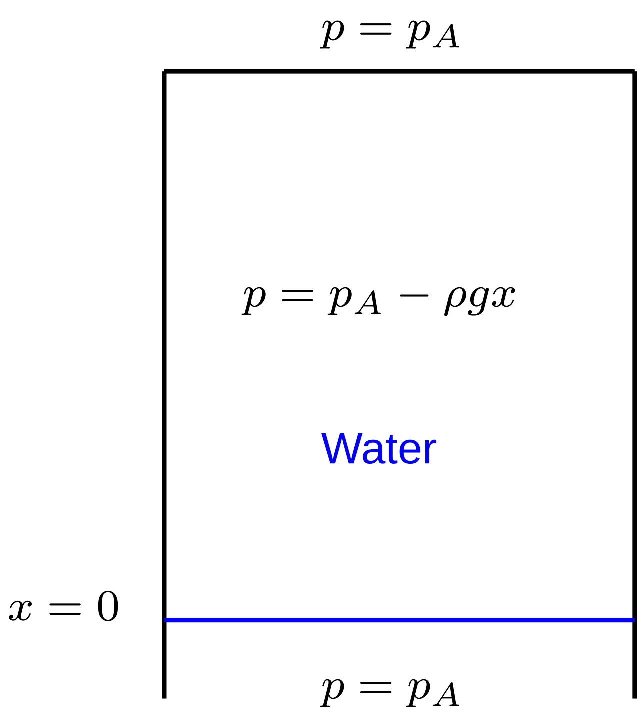
Low R_e flows: Slow, small and sticky. Equations are Linear. Solutions are unique and stable.

High R_e flows: Fast, vast, but don't last. Solutions may be nonunique and are often unstable.

My cup (sometimes) runneth over

A really simple Fluids problem: What happens if we invert a cup of water? To remain in the cup, somehow the body of water must be supported against gravity.

But this is easy to do if the cup has a solid bottom (and the column of water is less than 10m high). The pressure is atmospheric ($p = p_A$) everywhere except in the water, where it is lower: $p = p_A - \rho g x$



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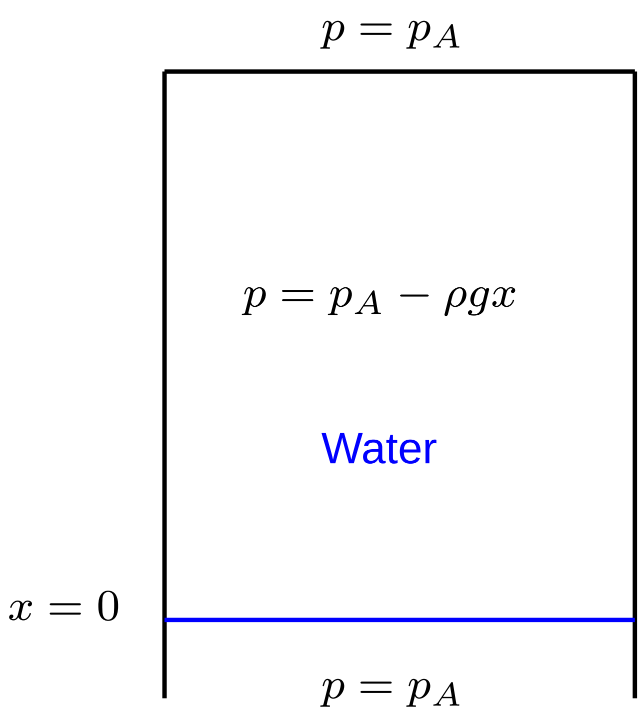


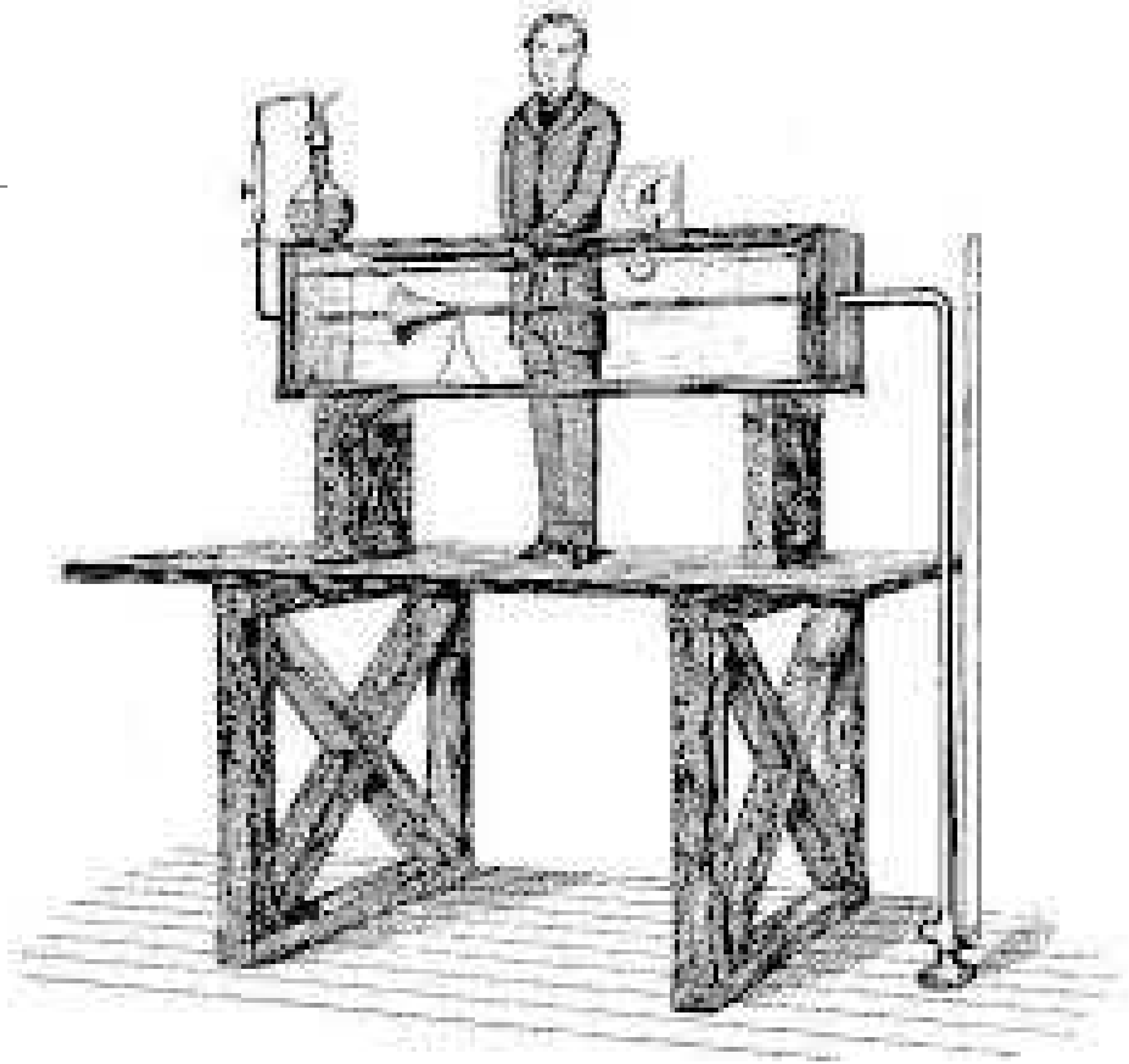
Let's try it! "Things done well, and with a care, exempt themselves from fear; Things done without example, in their issue are to be fear'd. **HENRY VIII Act 1 Scene 2**

Parallel flow down a pipe or duct

The simplest flow problem – a uniform pressure gradient $G \propto R_e^{-1}$ drives steady flow in the x-direction. Nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ vanishes and

$$p = -Gx \qquad \text{and in 2D} \quad u = 1 - y^2, \qquad \text{or in axi-symmetry,} \quad u = 1 - r^2$$

is a steady solution for all Reynolds number.
Osborne Reynolds' experiment showed that steady flow in a pipe becomes unsteady at large Reynolds number, and turbulent soon thereafter.



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Theoreticians perform stability analysis: Axisymmetry is a bit messy, so look at 2-D flow

$$\mathbf{u} = (1 - y^2)\widehat{\mathbf{x}} + \varepsilon \mathbf{u}_1(y) e^{ikx + imz + st}$$

As ε is arbitrarily small, we obtain a linear differential eigenvalue problem. For each mode (k, m) and R_e we find the growth rate s as an eigenvalue. $\Re e[s] > 0$ for $R_e > 5772$. The Numerics agree, as do careful experiments. It all makes physical sense; everyone is happy.

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Now do the same thing in axisymmetry, $(r,\,\theta,\,x)$ writing

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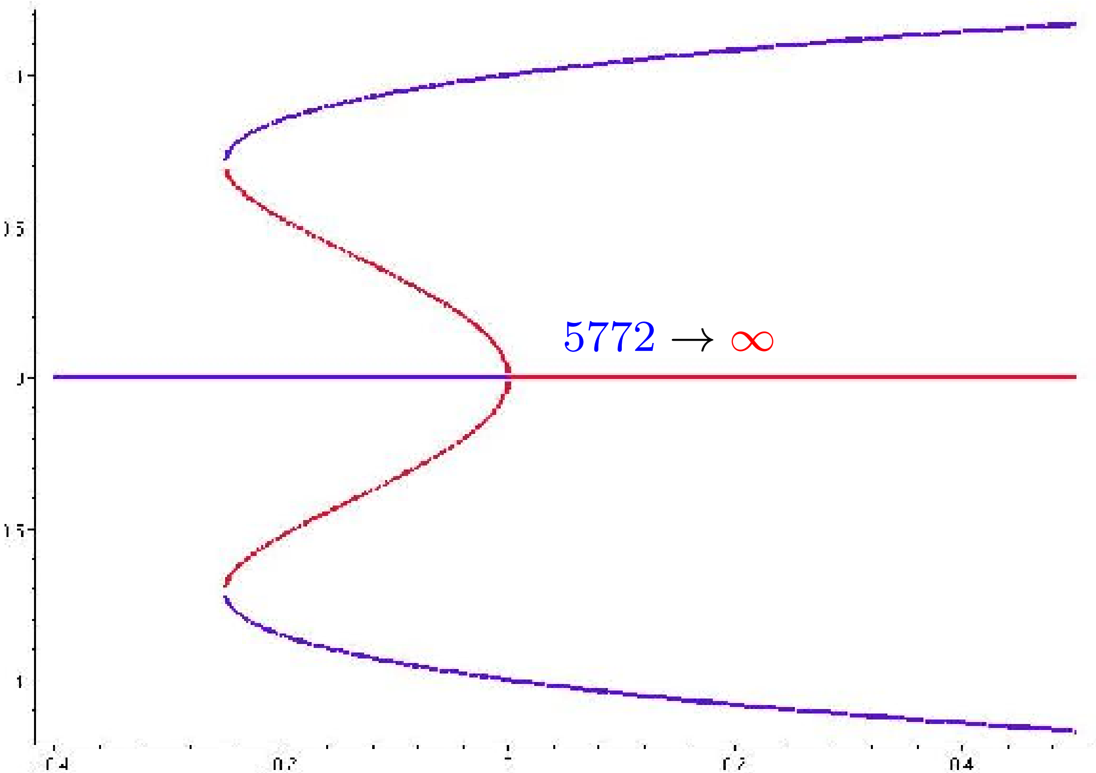
Heisenberg decides hydrodynamical stability is too difficult.

So what is going on?

Plane Poiseuille Flow has a subcritical bifurcation at $R_e \simeq 5772$, when infinitesimal disturbances become unstable. Below this value, larger disturbances are unstable, and head towards another solution.

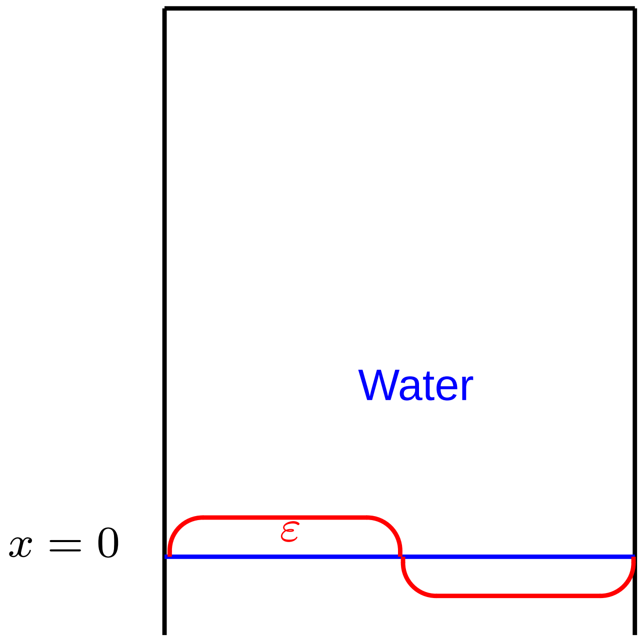


Pipe flow is indeed stable to all infinitesimal disturbances for all R_e; however the tolerance for ‘infinitesimal’ decreases with R_e. Formally, there is a subcritical bifurcation from infinity.

So what is going on?

What about the inverted glass? We should consider the stability of the equilibrium with $\mathbf{u} = 0$ and a surface at $x = 0$. Perturb the surface to

$$x = \varepsilon J_m(kr)e^{im\theta+st}, \qquad \text{where } J_m \text{ is a Bessel function.}$$



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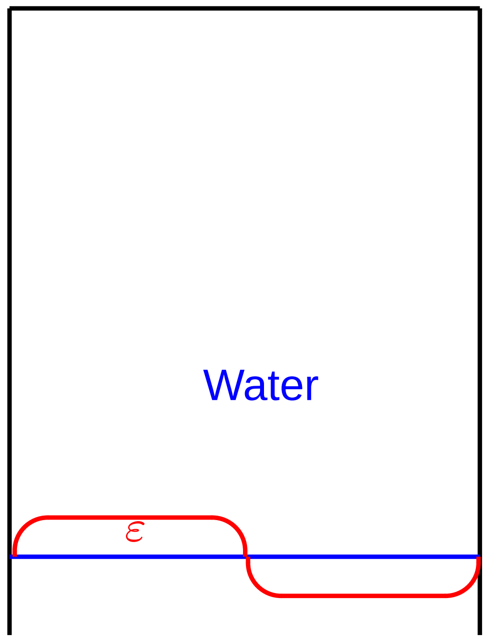
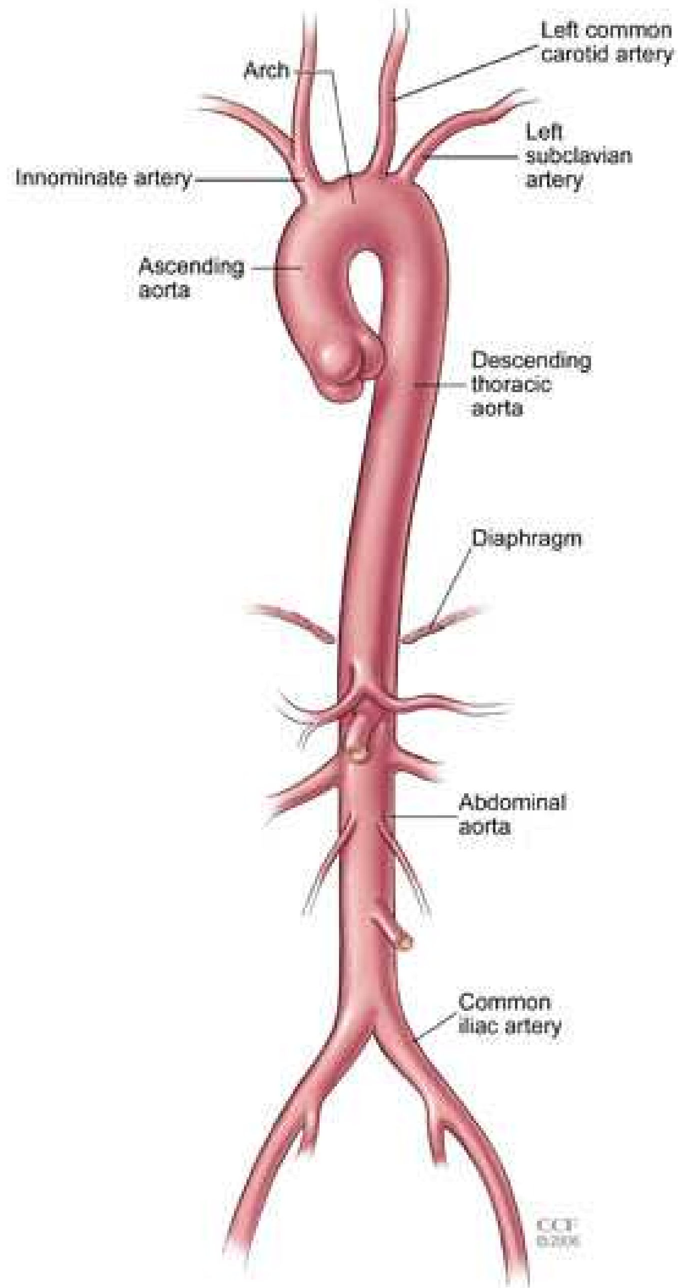


We find $s^2 = gk - \gamma k^3$, where γ represents surface tension and k satisfies $J'_m(ka) = 0$. Hopelessly unstable for radius $a > 2cm$. What we saw could not happen and is unrepeatable – the Real World is capricious.

Real pipes can have bloody geometry



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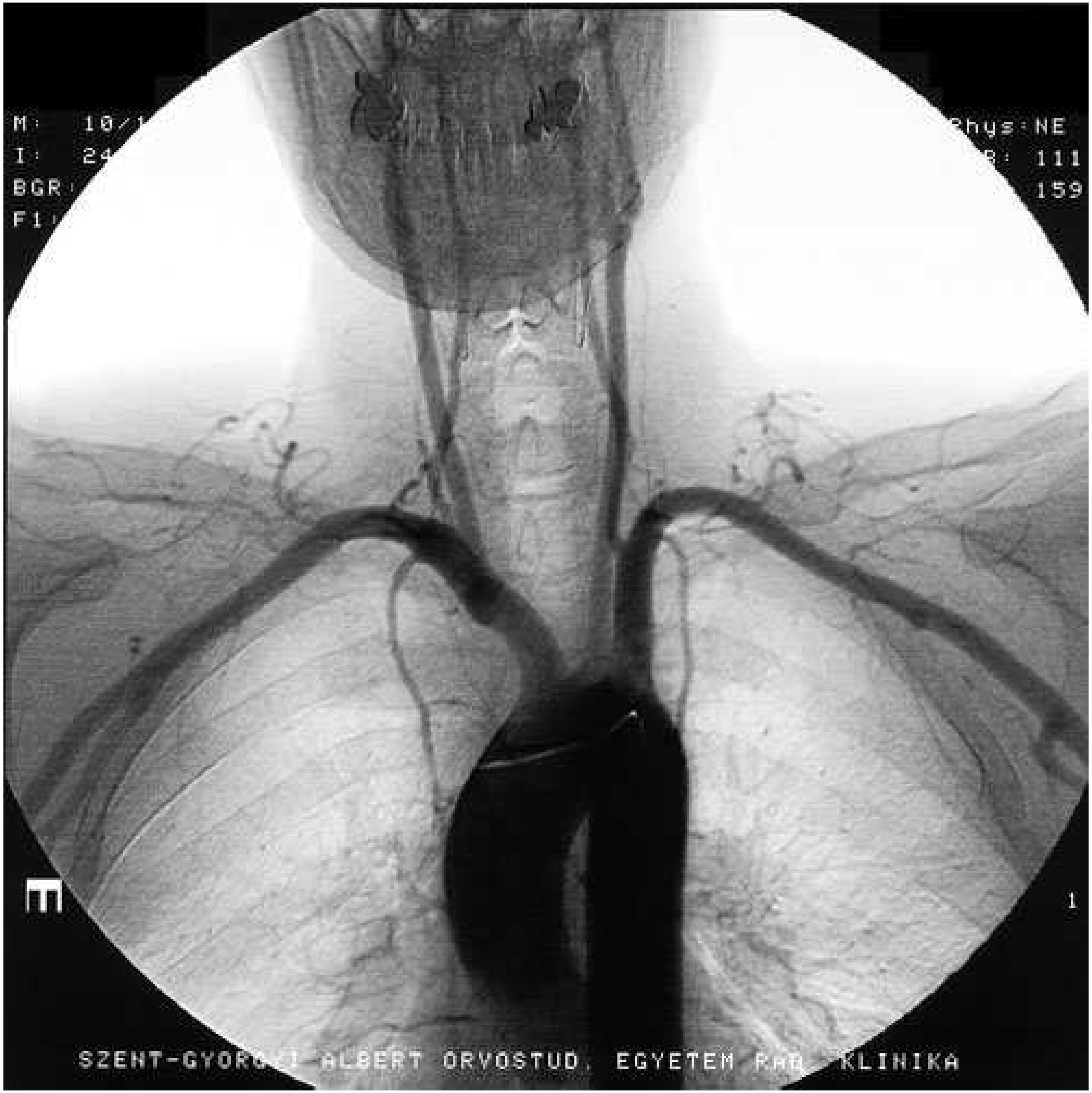

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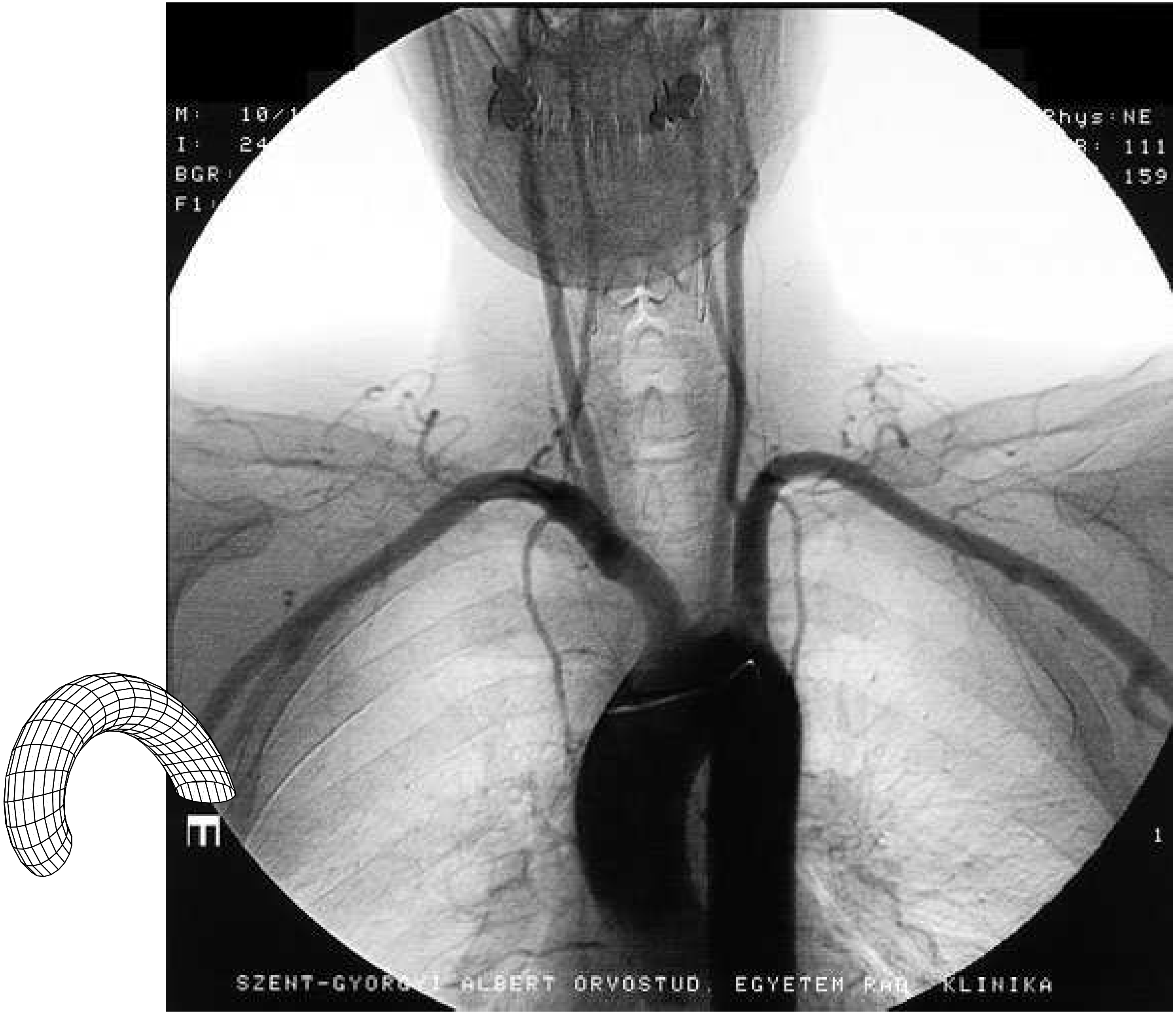


Some arteries are helical – the umbilical cord

Slightly curved pipes – The Dean Equations

Let's examine a simpler curved pipe in some depth.

Consider steady flow down a toroidal pipe which curves slowly and uniformly.
After some simplification, the governing equations reduce to

$$\begin{aligned} \mathbf{u}\cdot\nabla w \equiv J(\psi, w) &= 4 + \nabla^2 w \\ \mathbf{u}\cdot\nabla \Omega \equiv J(\psi, \Omega) &= \nabla^2\Omega + 576Kww_z \\ \text{where} \qquad \Omega &= +\nabla^2\psi \end{aligned}$$

These equations are to be solved for the down-pipe velocity $w(x, z)$ and cross-pipe streamfunction $\psi(x, z)$ subject to the no-slip conditions

$$\nabla\psi = 0, \qquad w = 0 \qquad \text{on the pipe boundary.}$$

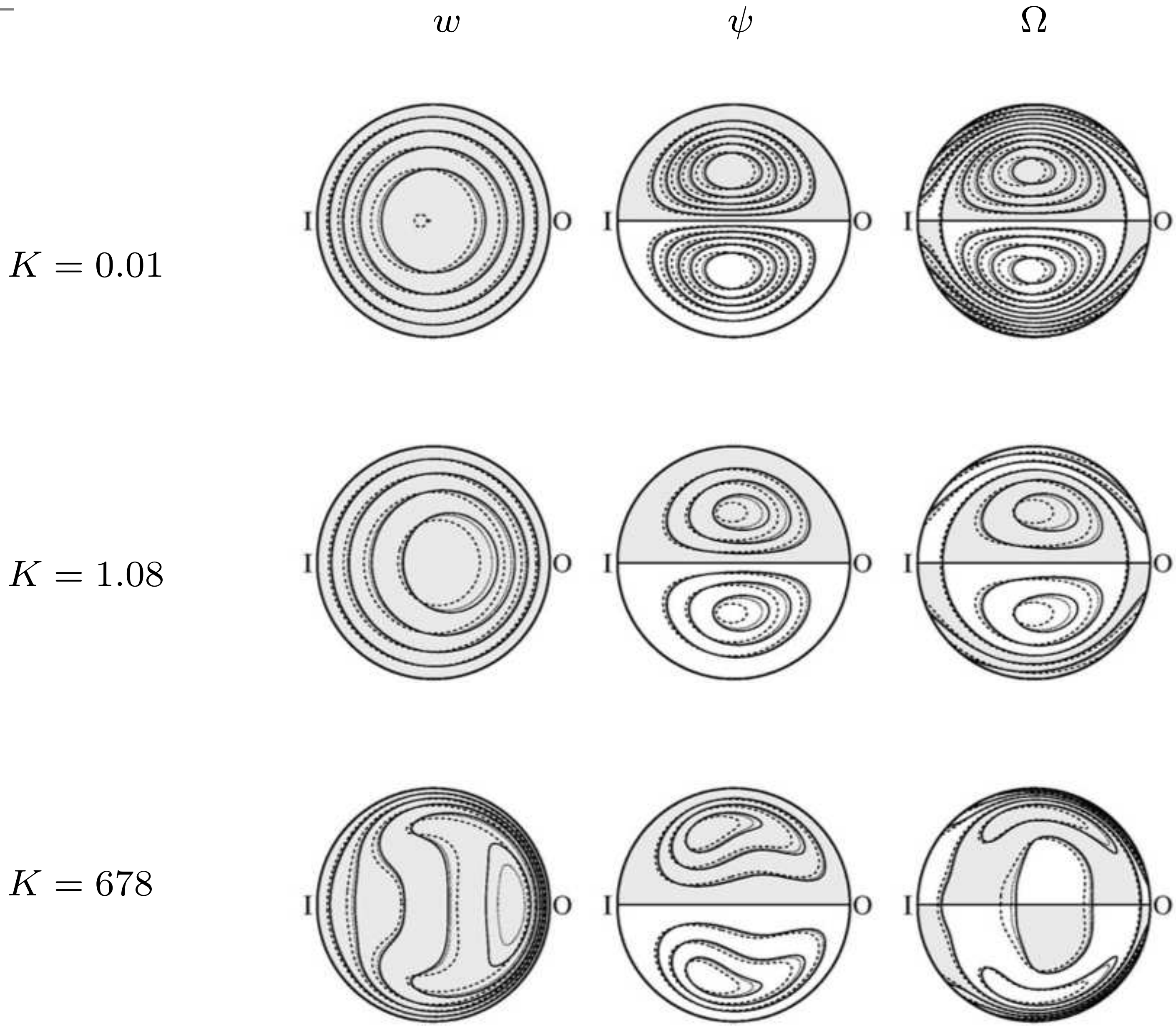
There is one parameter in the problem, K, which is known as the Dean number. It is a Reynolds number modified by the toroidal pipe curvature.

The centrifugal gradient drives a secondary flow, characteristically with two vortices, known as the Dean vortices.

Known solutions in a circle

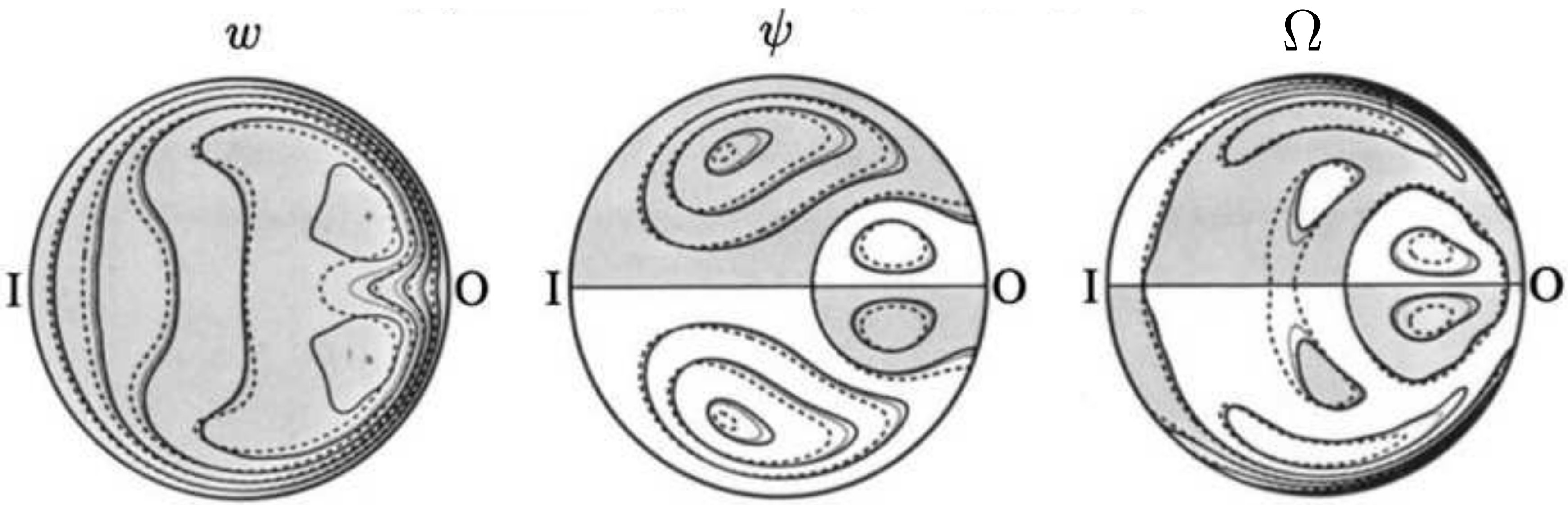
As K increases, the solution has a number of key features:

Known solutions in a circle



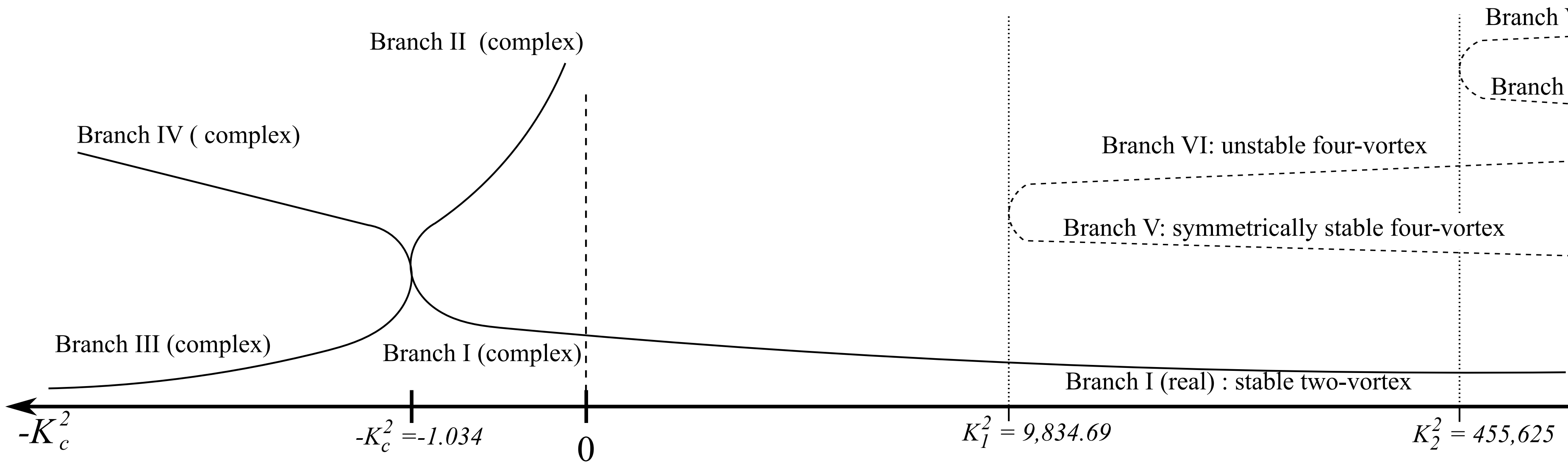
Known solutions in a circle

Bifurcation to “4-vortex" flow at high-K (from Siggers & Waters)



Known solutions in a circle

Schematic of bifurcation structure for steady flows.:



[Warning: Practically no Fluid Mechanics from here on. Henceforth we will consider how to extract information from the first N terms of an infinite series.]

Series representation for small K

At $K = 0$ we have pipe Poiseuille flow

$$w_0 = 1 - r^2, \qquad \psi_0 = 0$$

Can seek an expansion

$$w = \sum_{n=0}^{\infty} K^n w_n(r,\,\theta), \qquad \psi = \sum_{n=0}^{\infty} K^n \psi_n(r,\,\theta).$$

At each order we must solve linear problems of the form

$$\nabla^2 w_n \sim \sum_p \sum_q A_{npq} r^p \cos(q\theta) \qquad \nabla^4 \psi_n \sim \sum_p \sum_q B_{npq} r^p \cos(q\theta)$$

(or $\sin q\theta$ as appropriate) with $\psi_n = \psi_n' = v = 0$ on $r = 1$. These can be solved exactly to all orders, giving for example

$$\psi_1 = \left(4r - 9r^3 + 6r^5 - r^7\right)\cos\theta$$

$$w_1 = \left(\tfrac{19}{40} r - r^3 + \tfrac{3}{4} r^5 - \tfrac{1}{4} r^7 + \tfrac{1}{40} r^9\right)\sin\theta.$$

Where is the series valid?

We have an infinite series representation of the solution for any K.

BUT the series has finite Radius of Convergence, and stops converging before exhibiting any bifurcations or asymptotic structure. Nevertheless, we can learn about the solution outside the circle of convergence using various series extension techniques.

Firstly, we can locate the singularity which limits convergence by looking at the ratio of adjacent terms, in a Domb-Sykes plot.

For convenience, restrict attention to one particular flow feature, e.g. the flux down the pipe

$$Q(K) = \frac{2}{\pi} \int w \, dA = \sum_{n=0}^{\infty} a_n K^{2n}$$

Suppose

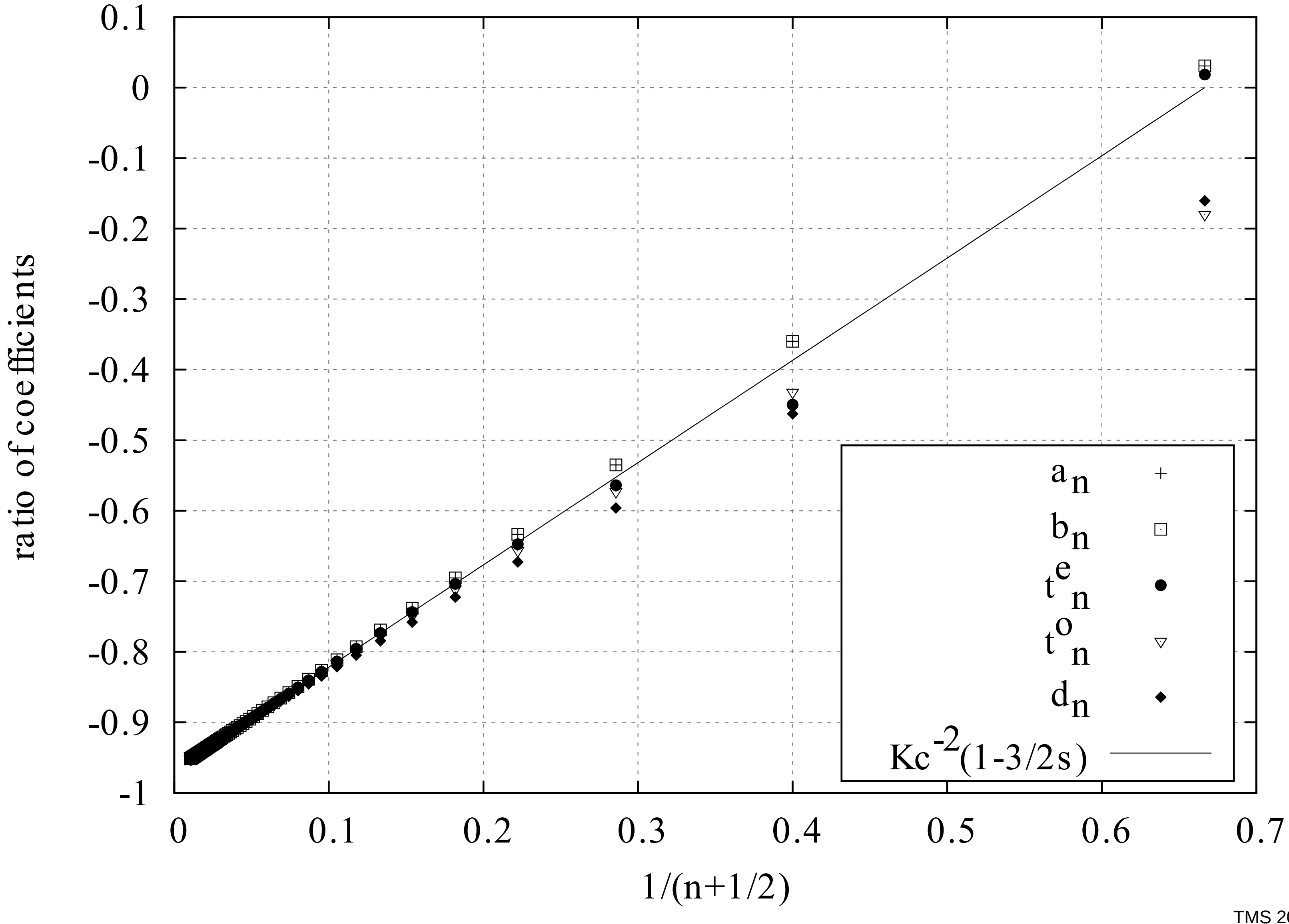
$$Q = (K^2 + K_c^2)^{\alpha} f(K) + g(K)$$

where f and g are analytic for $|K| \leqslant K_c$. Can show

$$\frac{a_{n+1}}{a_n} = -\frac{1}{K_c^2} \left[1 - \frac{\alpha+1}{n} + O\left(\frac{1}{n^2}\right) \right]$$

Where is the series valid?

The data suggest an (imaginary) singularity of this form with $\alpha = 1/2$:



Euler transformation – attempted analytic continuation

Boshier calculated terms up to K^{196} (Van Dyke had 24 in 1974). We can locate the singularity extremely accurately:

$$K_c = 1.016994400993538278241609624866535081135717913624 23\ldots$$

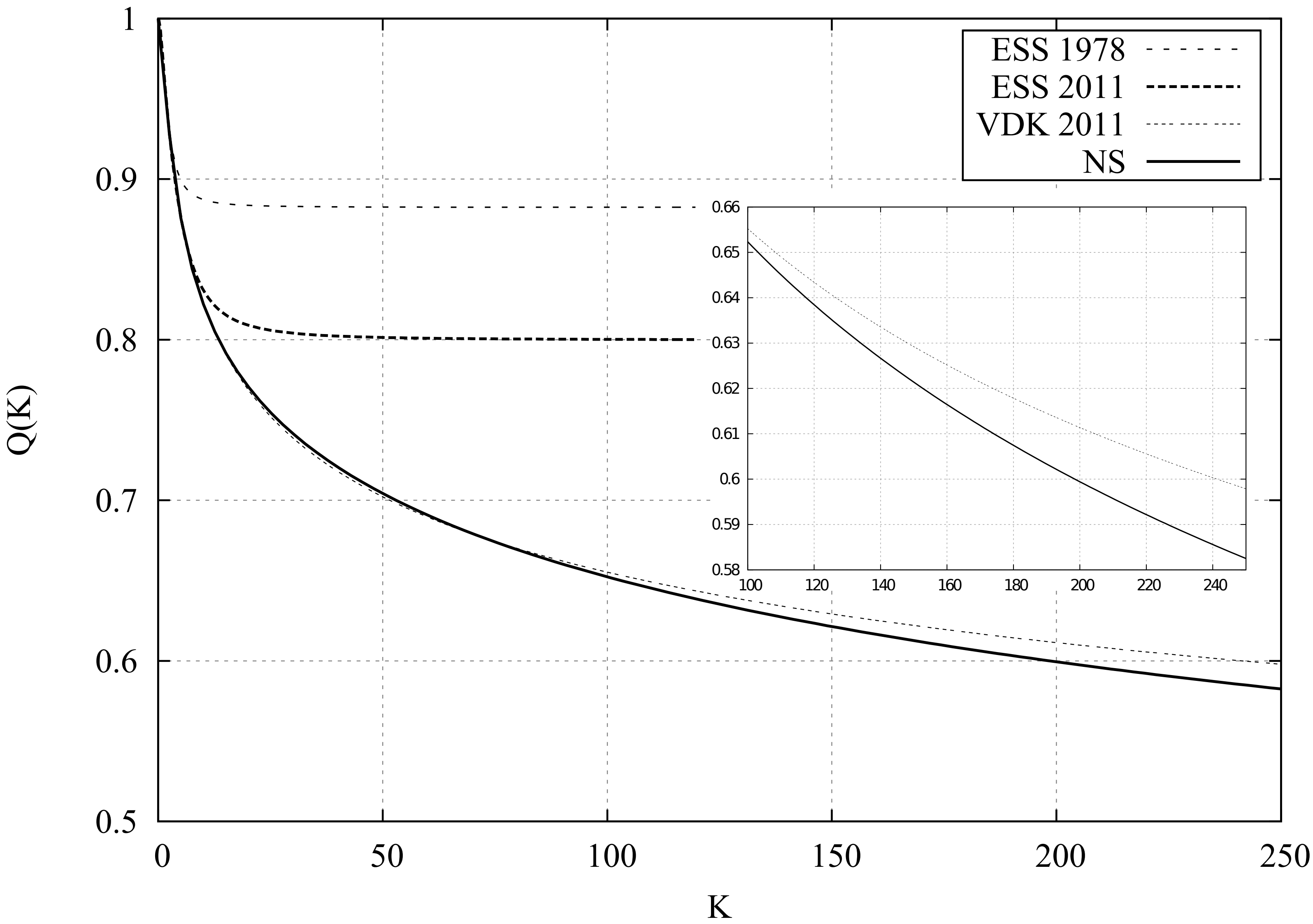
To learn about the behaviour for large K, Van Dyke suggested the Euler transformation

$$\varepsilon = \frac{K^2}{K^2 + K_c^2}$$

so that $K^2 = -K_c^2$ goes to $\varepsilon = \infty$, while $K = \infty$ maps to $\varepsilon = 1$. Can then expand the new series in ε, and try to analyse its behaviour.

We find that the new ε-series appears to converge in $|\varepsilon| < 1$. However the singularity at $\varepsilon = 1$ is difficult to identify reliably.

Euler transformation – attempted analytic continuation



Generalised Padé Approximants à la Drazin & Tourigny

The direct series extension methods have limited success partly because the new series loses its arbitrary accuracy, but especially because the confluent singularities at infinity are very hard to separate out. (Very likely, infinity is an essential singularity.)

Instead, we use a scheme of Generalised Padé approximants, following Drazin & Tourigny for a different problem.

A simple Padé approximant would replace the polynomial approximant for $Q(K)$ by a rational function

$$Q(K) = \frac{f(K)}{g(K)} = \frac{f_0 + f_1 K^2 + f_2 K^4 + \ldots f_L K^{2L}}{g_0 + g_1 K^2 + g_2 K^4 + \ldots f_M K^{2M}} \simeq \sum_{n=0}^{N} a_n K^{2n}$$

Any poles in $Q(K)$ can be represented as zeros of $g(K)$.
However this doesn't work so well with branch cuts such as $(K^2 + K_c^2)^{1/2}$.

To obtain the square root, we could seek polynomials f, g, h such that the series for $Q(K)$ is a root of

$$f(K)Q^2 + g(K)Q + h(K) = 0.$$

Interestingly, this equation has more than one branch.

Generalised Padé Approximants à la Drazin & Tourigny

More generally, we can define a d-th order approximating formula equivalent to order K^N for Q.

$$0 = F_d(K,Q) \equiv \sum_{n=1}^{d} \sum_{m=1}^{n} f_{n-m,n} K^{n-m} Q^n = 0.$$

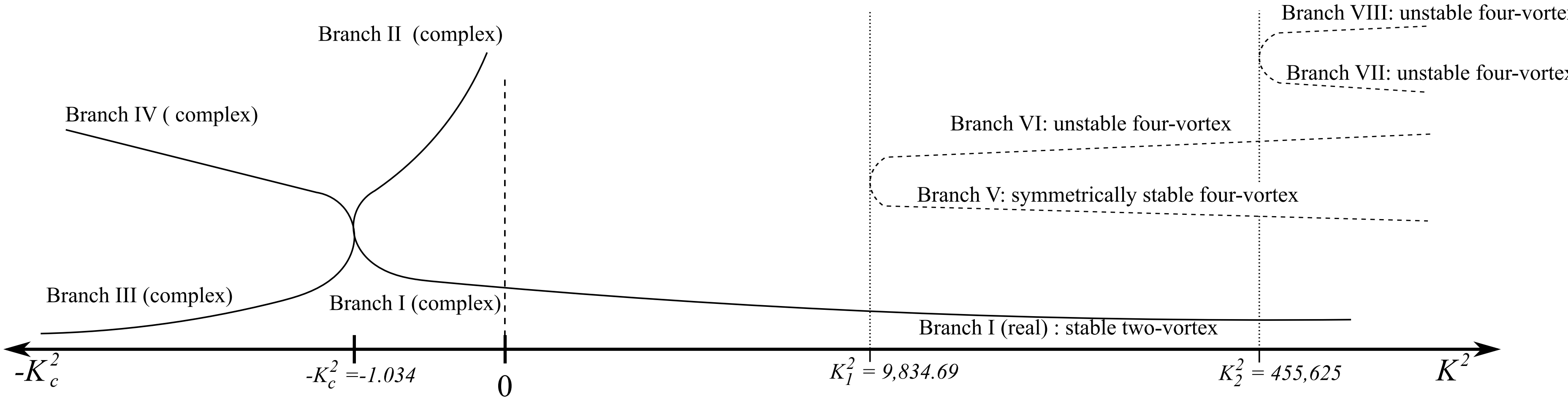
The coefficients are uniquely determined for $2N = d^2 + 3d - 2$.

For each d, we can plot all branches of $Q(K)$. One of these corresponds to our earlier solution. Some branches are spurious, but others persist as d increases.

These robust alternative branches probably correspond to other solutions, bifurcating off the main branch.

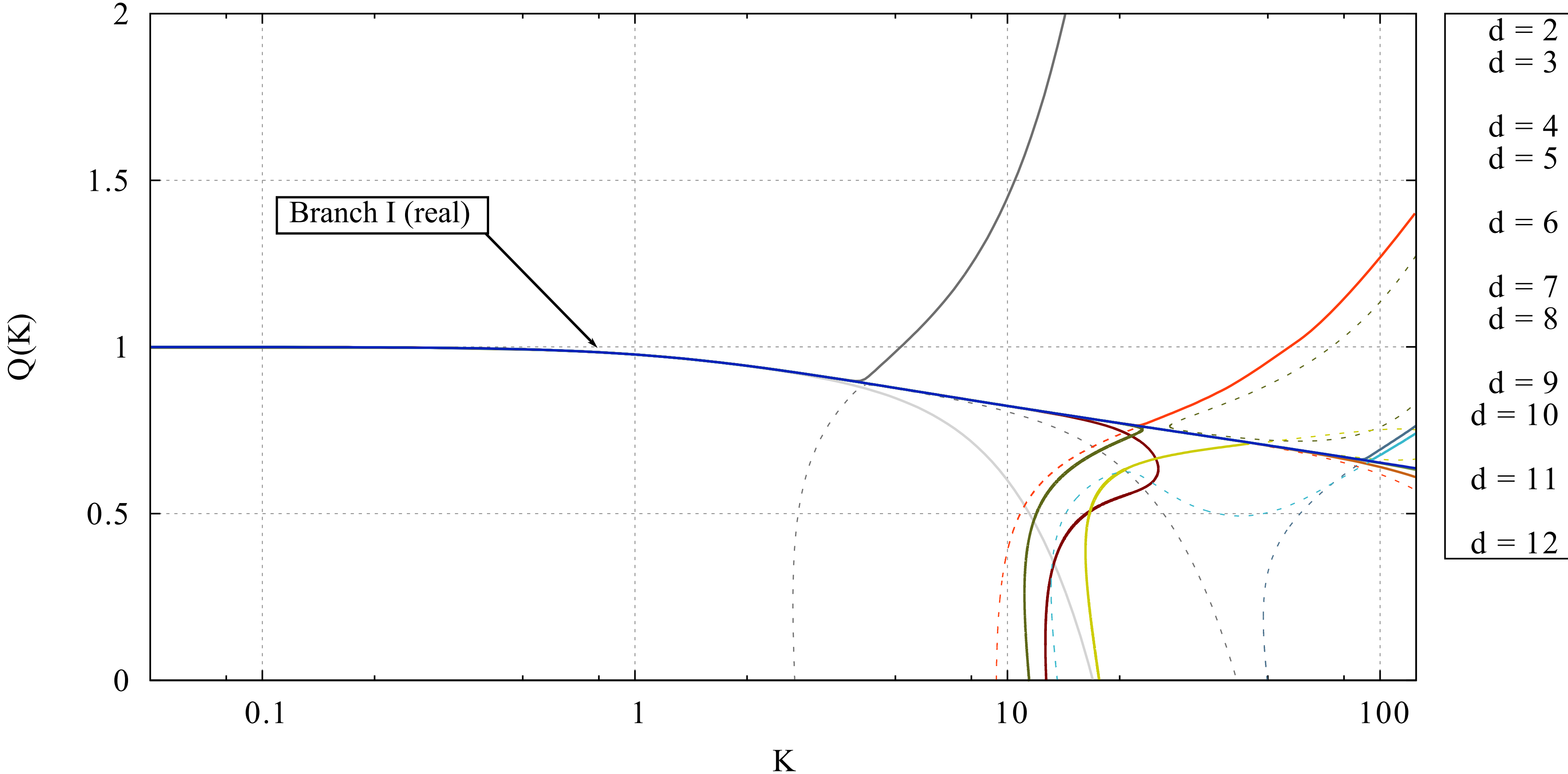
Generalised Padé Approximants à la Drazin & Tourigny

Recall the schematic



Generalised Padé Approximants à la Drazin & Tourigny

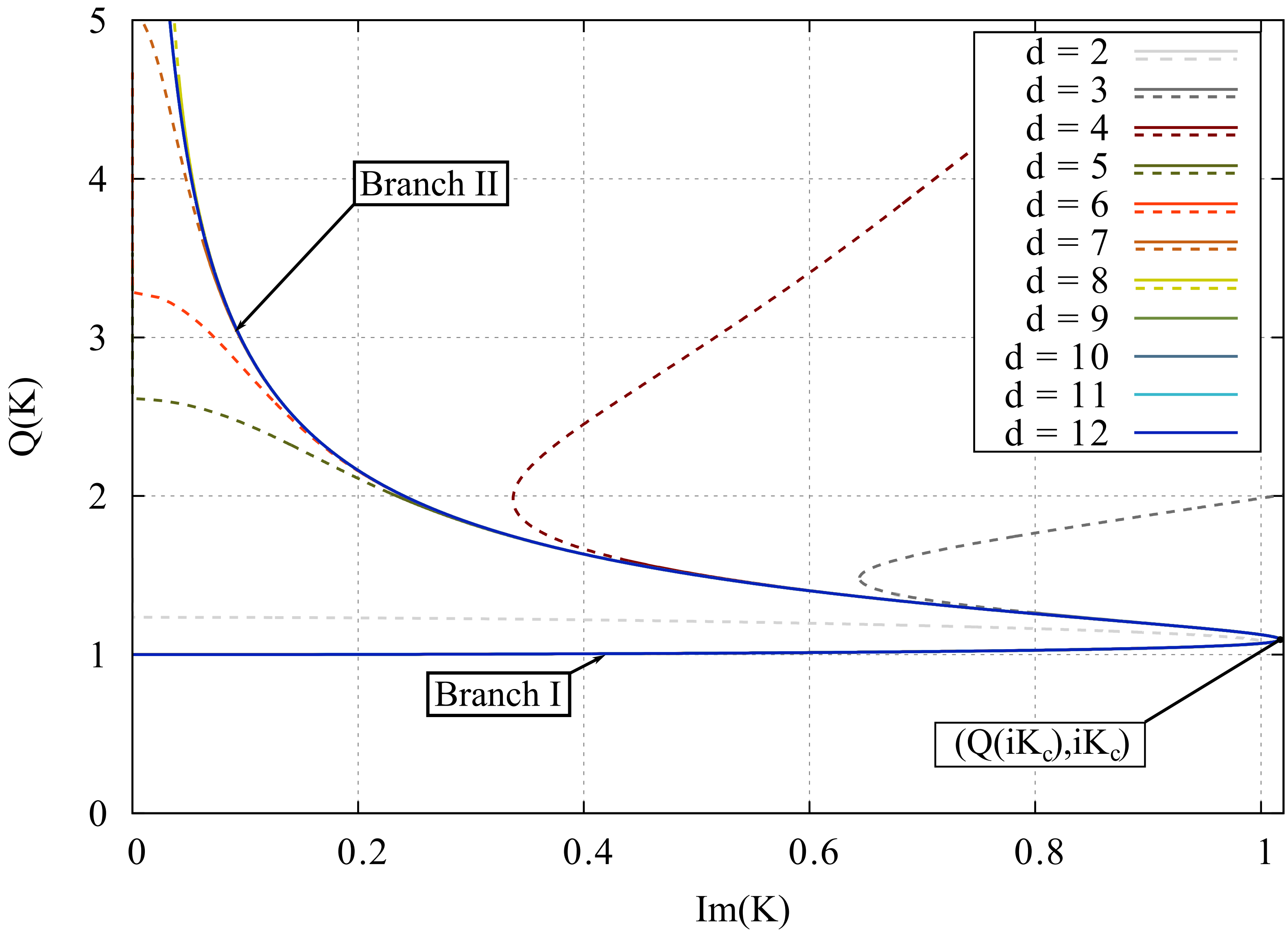
Here are the approximants for the main branch for $0 < K < 150$



Note we do NOT see the 4-vortex solution. This suggests that the bifurcation does not occur at finite K, but rather off $K = \infty$ (compare stability of pipe Poiseuille flow.)

Generalised Padé Approximants à la Drazin & Tourigny

The square-root singularity at $K = \pm iK_c$ also appears

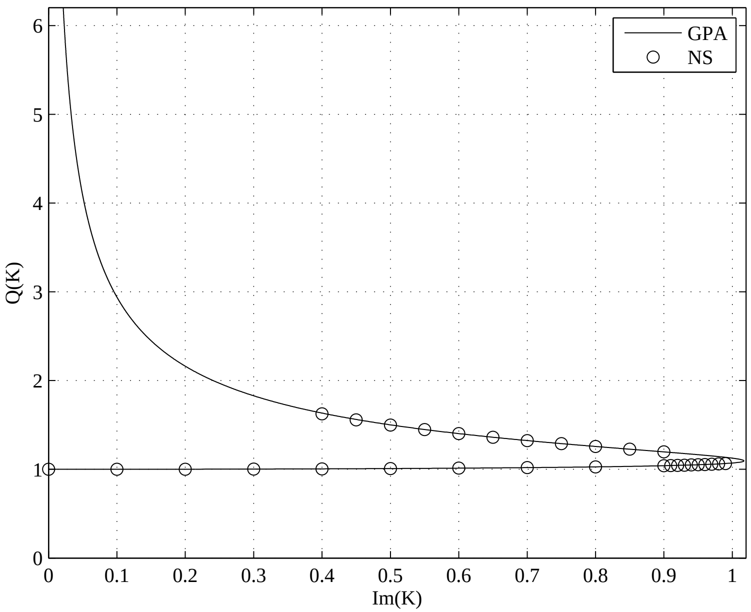


Comparison with Navier-Stokes Numerics

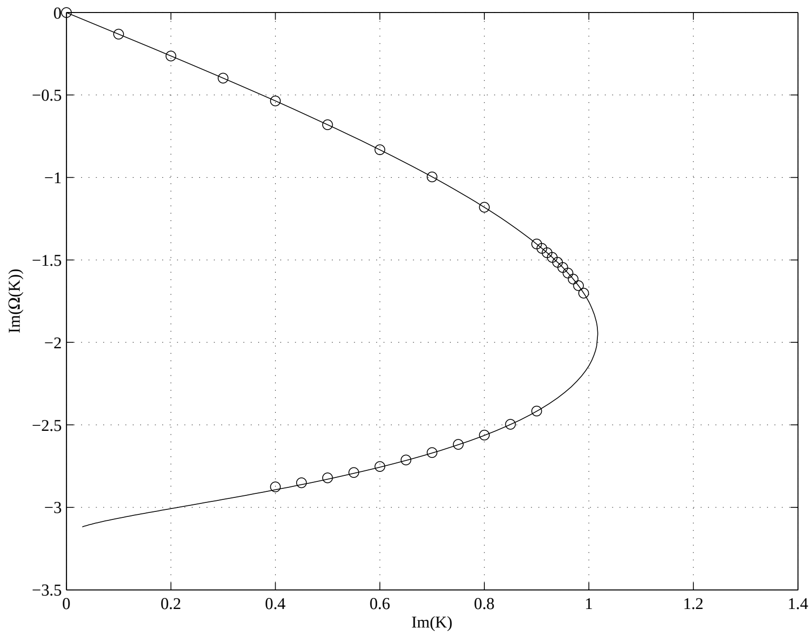
Do the robust GPA-branches for $Q(K)$ correspond to real solutions?
Seek steady solutions of the Navier-Stokes using a spectral decomposition in θ and finite differences in r. Use (pseudo-)path continuation to follow solution branches.

The real solutions agree well with those of other authors. We can also seek solutions for complex Dean number! They too agree with the Padé approximants.

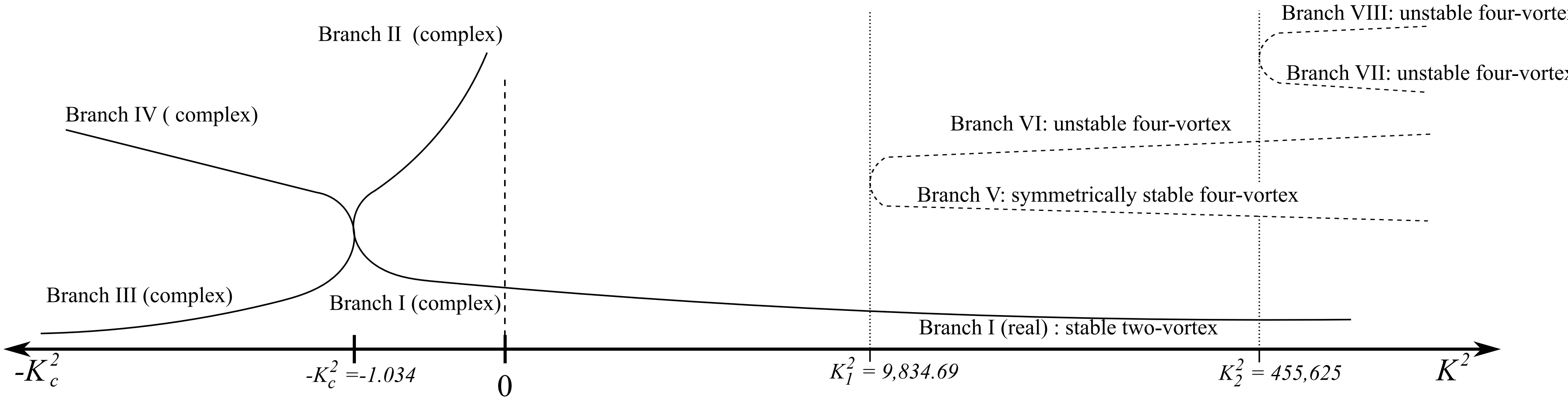
Flux



Total vorticity



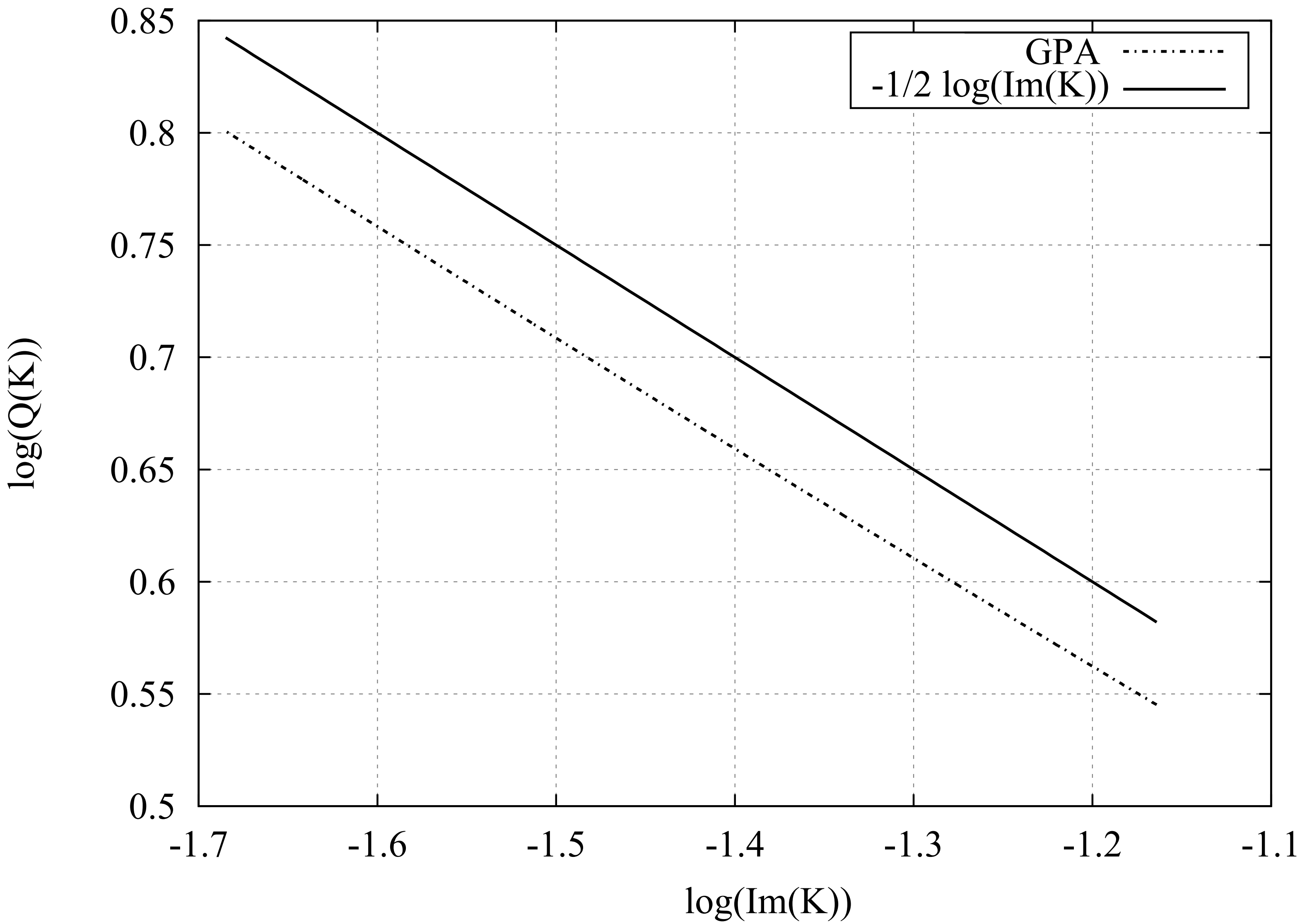
The other solution branch near $K = 0$



For $-K_c^2 < K^2 < 0$, there are two complex solutions with real flux. As $K \to 0$, on the upper branch $Q \to \infty$.The log-log graph suggests $Q(K) \sim K^{-1/2}$.

So despite the uniqueness theorem, are zero-Reynolds-number flows really unique? There are two solutions on the circle $|K| = \varepsilon$, for all $1 \gg \varepsilon > 0$

The other solution branch near $K = 0$



The other solution branch near $K = 0$

Does this make sense? The governing equations are

$$J(\psi, w) = 4 + \nabla^2 w, \qquad J(\psi, \nabla^2\psi) = Kww_z + \nabla^4\psi$$

We can rescale the problem with $w = K^{-1/2}W$ keeping ψ fixed to obtain

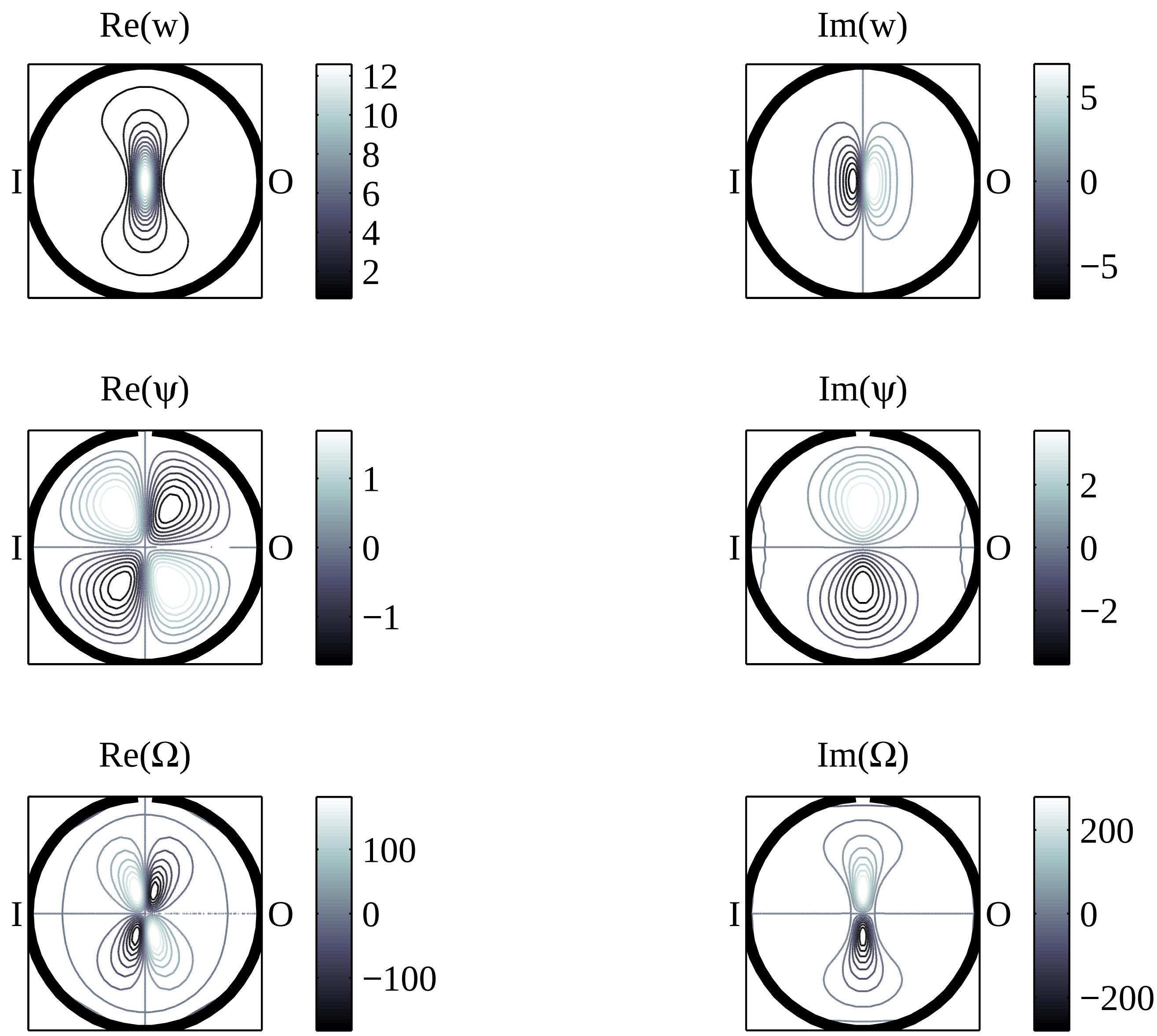
$$J(\psi, W) = 4K^{1/2} + \nabla^2 W, \qquad J(\psi, \nabla^2\psi) = WW_z + \nabla^4\psi$$

As $|K| \to 0$, as well as the ordinary Poiseuille flow solution, we have a complex solution to the unforced equations, a sort of eigenfunction.

$$J(\psi, W) = \nabla^2 W, \qquad J(\psi, \nabla^2\psi) = WW_z + \nabla^4\psi.$$

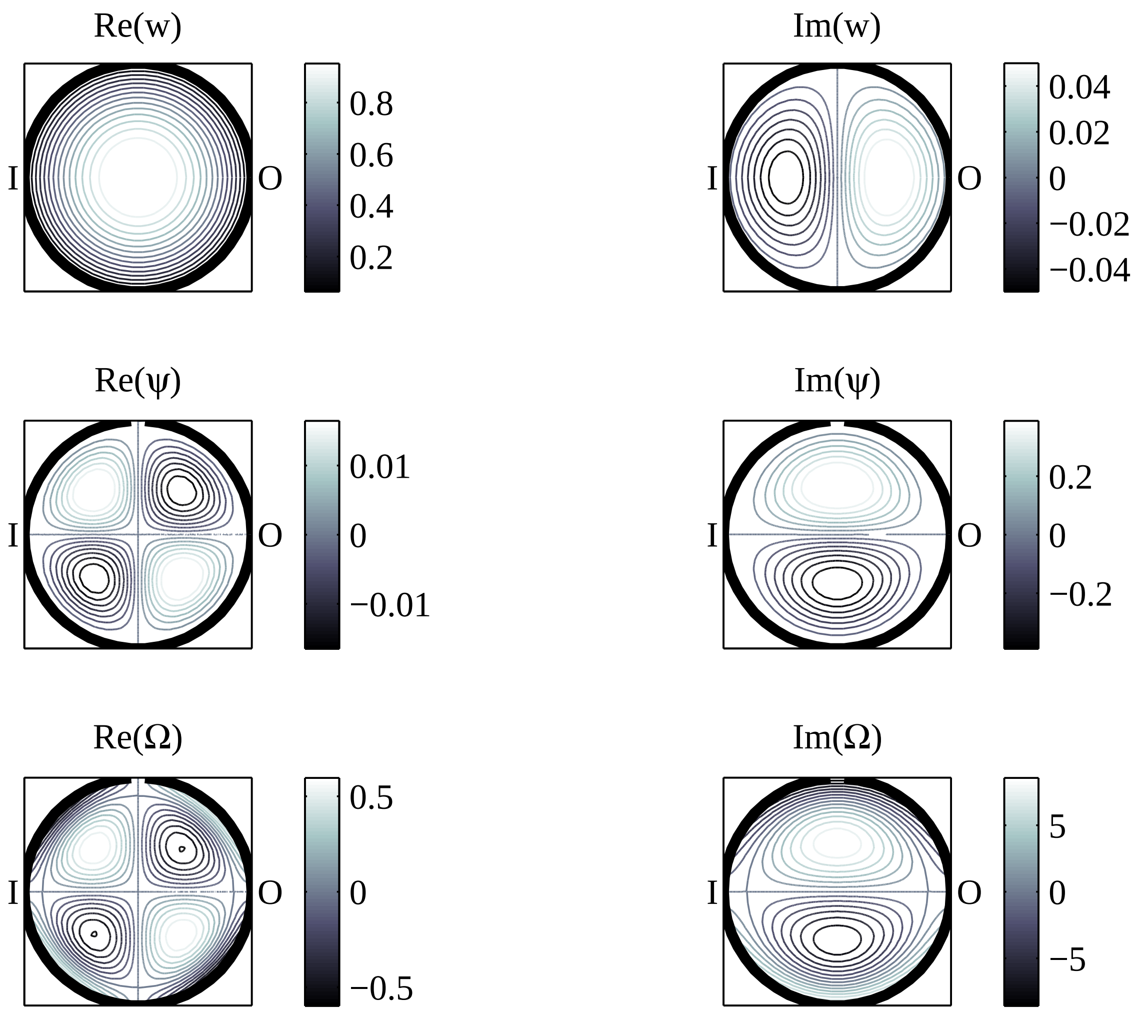
What does it look like?

The other solution branch near $K = 0$



The other solution branch near $K = 0$

For comparison, the solution on the lower complex branch I is



Concluding Remarks

- “Fluids" is fun, and fluids are fun. There is no question about it!
- Flows for which the zero-Reynolds-number solution is readily obtainable, may be represented by an infinite Stokes series.
- The convergence of this series tends to be limited by a singularity at imaginary Reynolds number.
- Analytic Continuation of this series is best approached by the Drazin & Tourigny method of Generalised Padé Approximants. This method finds solution branches, stable or not, which bifurcate off the main branch.
- Our belief is that the “4-vortex" Dean flows do not bifurcate off the main Dean-series branch.
- At small but imaginary Reynolds numbers there tend to be two (complex) solutions, which persist until $R_e = 0$, although the effective Reynolds number is actually $O(1)$ then, as the velocity becomes infinite.

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- So Zero-Reynolds-Number flows ARE REALLY unique. But they may not be COMPLEXLY unique. (Boshier & Mestel, JFM 2014)
- And let Euler have the final word:

Concluding Remarks

Leonhard Euler (1775)

Principia pro motu sanguinis per arterias determinando

14. Ecce igitur ambae nostrae aequationes, quas nobis principia continuitatis et accelerationis suggesserunt, ita se habebunt

$$\text{I.} \quad \left(\frac{ds}{dt}\right) + \left(\frac{d \cdot vs}{dz}\right) = 0,$$

$$\text{II.} \quad 2g\left(\frac{dp}{dz}\right) + v\left(\frac{dv}{dz}\right) + \left(\frac{dv}{dt}\right) = 0,$$

ex quibus ergo si ipsis adiungatur formula

43. In motu igitur sanguinis explicando easdem offendimus insuperabiles difficultates, quae nos impediunt omnia plane opera Creatoris accuratius perscrutari; ubi perpetuo multo magis summam sapientiam cum omnipotentia coniunctam admirari ac venerari debemus, cum ne summum quidem ingenium humanum vel levissimae vibrillae veram structuram percipere atque explicare valeat.