Stretching, bending, twisting and coiling: how to build a fluid-mechanical sewing machine

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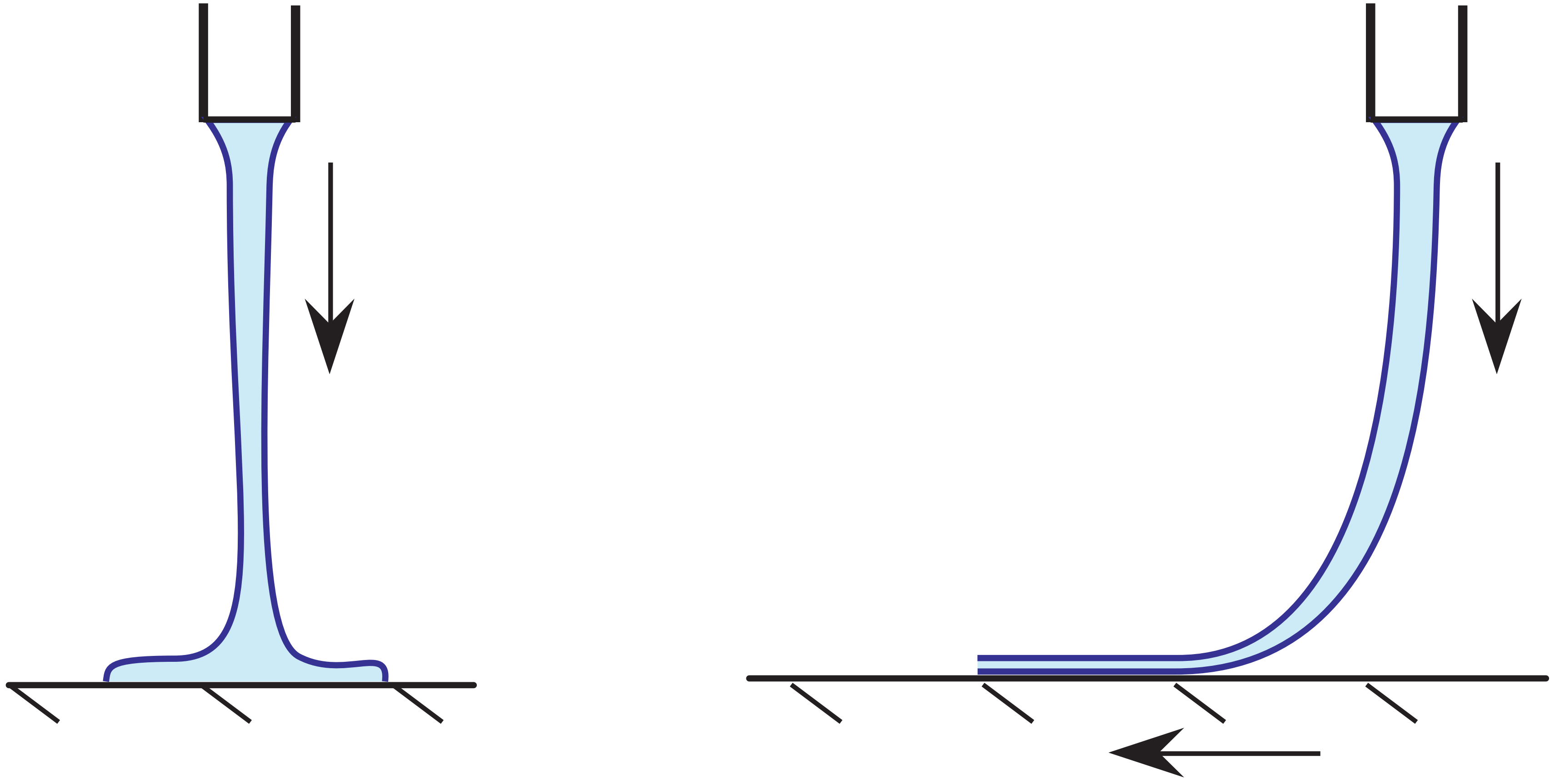
ITG, DAMTP

Trinity Mathematical Society, 6 February 2012

Something Simple for Breakfast?

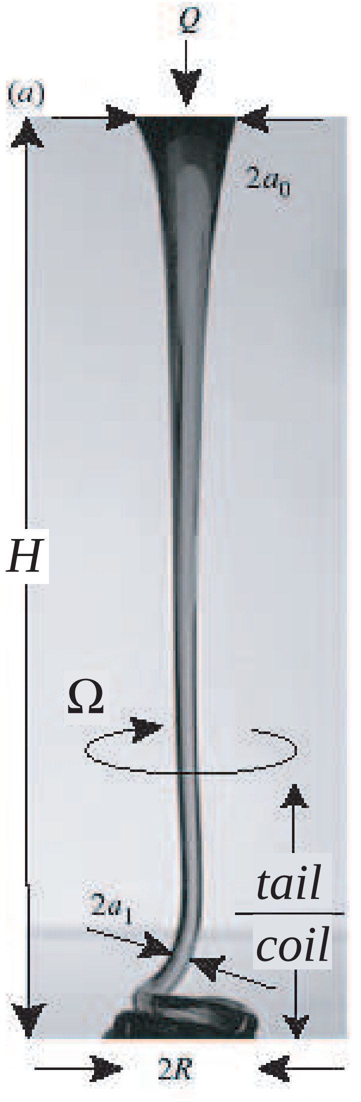
A thread of viscous fluid falling on a surface



Coiling on a Stationary Surface

- ∃ Steady coiling solution (helical wave)

Mahadevan et al. 2000



Ribe 2004 ⇐ ⇐

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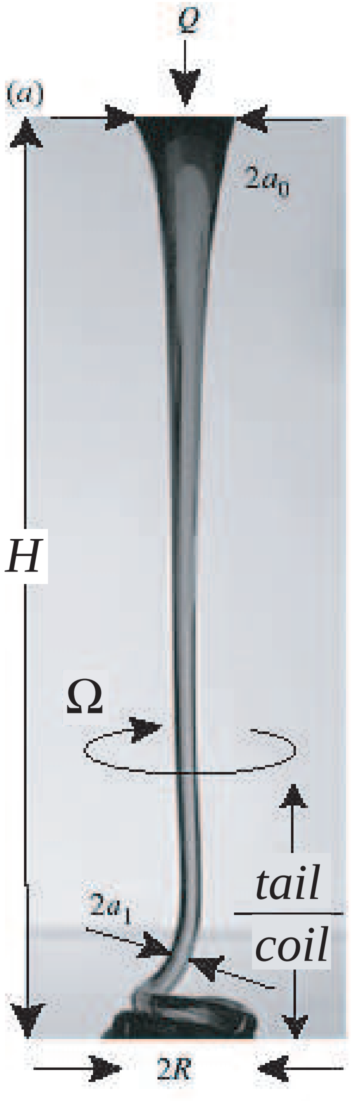

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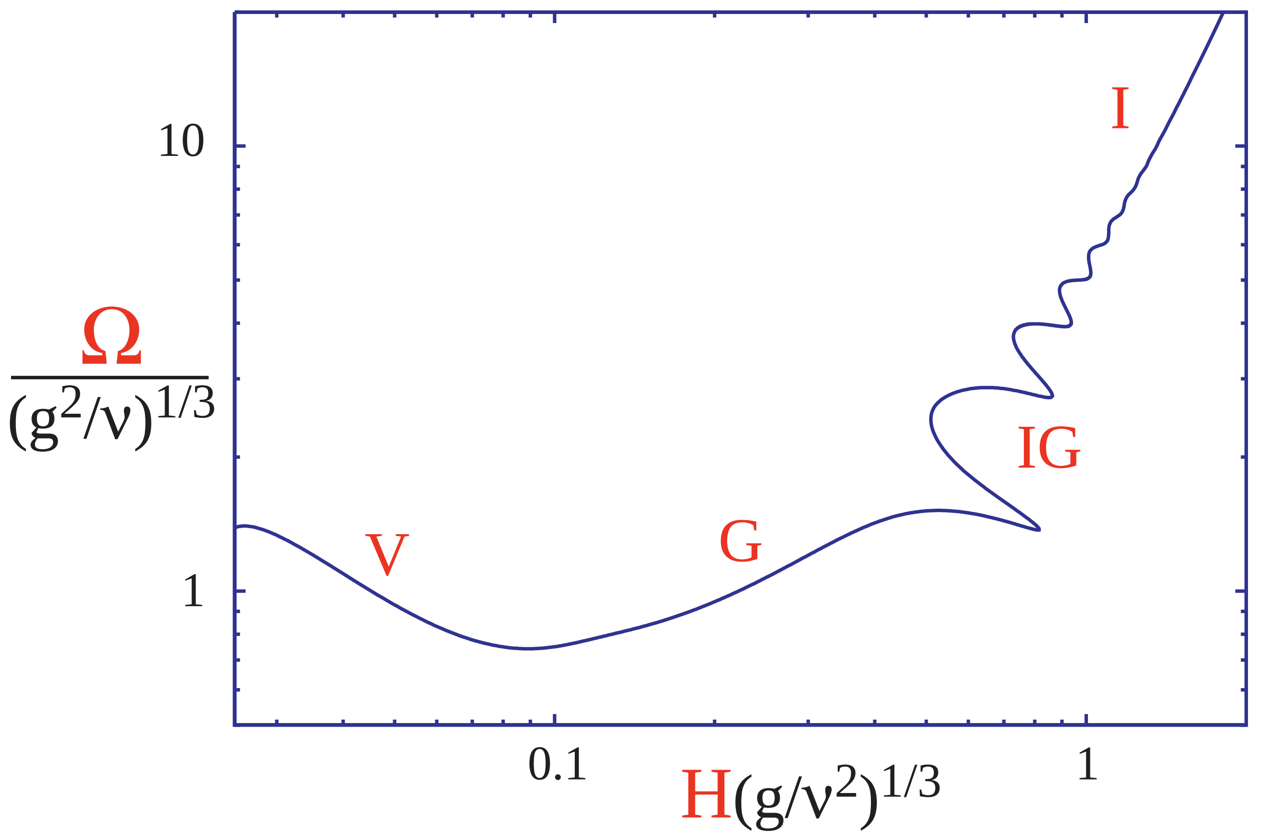
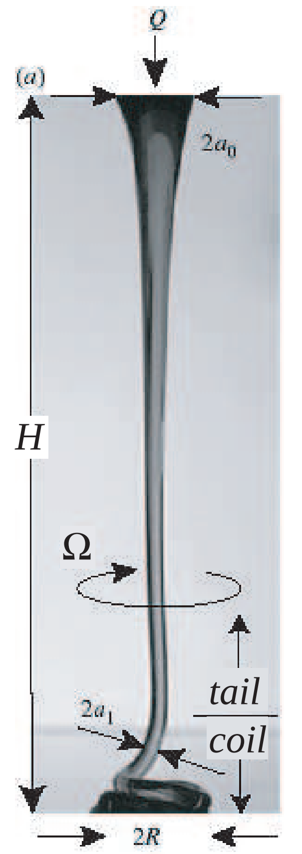
Not So Simple!





What determines the frequency?

What happens if the surface is moving?

On Board

Stress linear in velocity gradients. Translation and rotation irrelevant

Viscous stress in simple shear

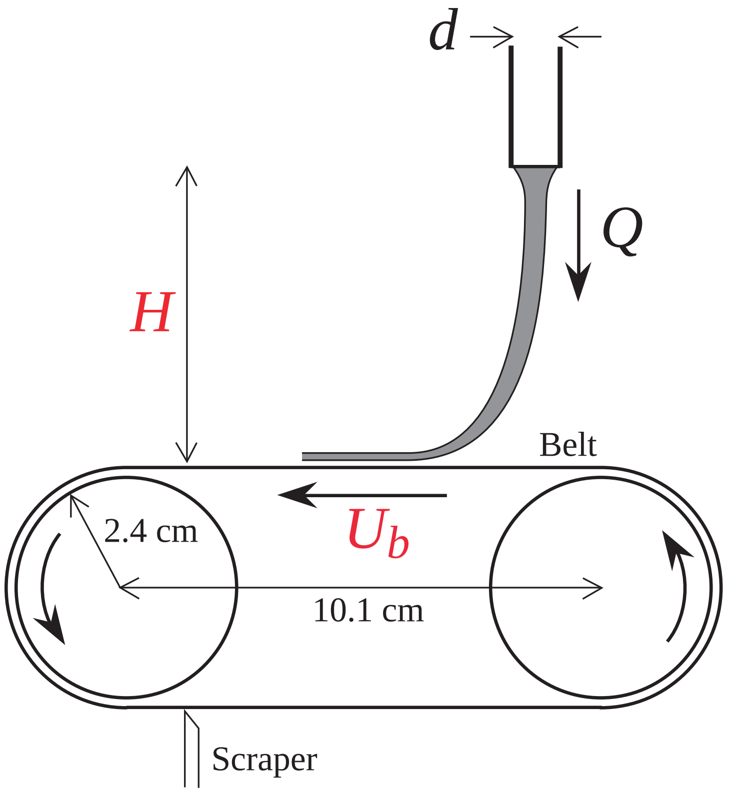
Viscous stress in axial extension

Mass conservation for a thread

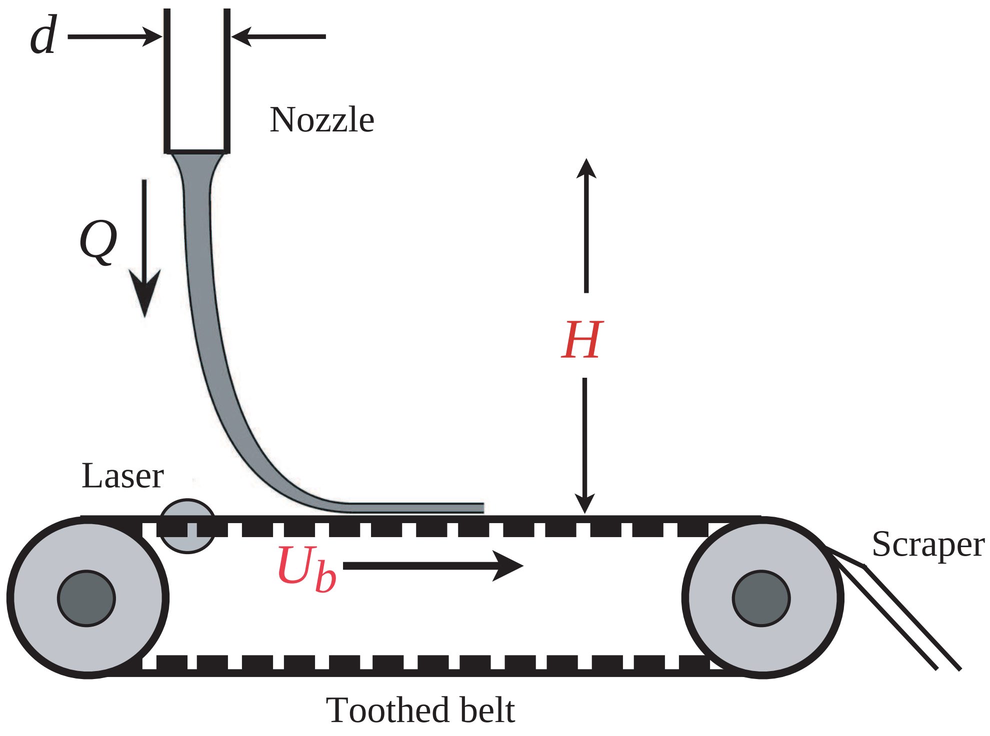
Steady vertical fall solution - infinite fall, finite fall.

Bending

Fall onto a Moving Belt



Chiu-Webster & Lister 2006



Morris, Ribe, Dawes & Lister 2008

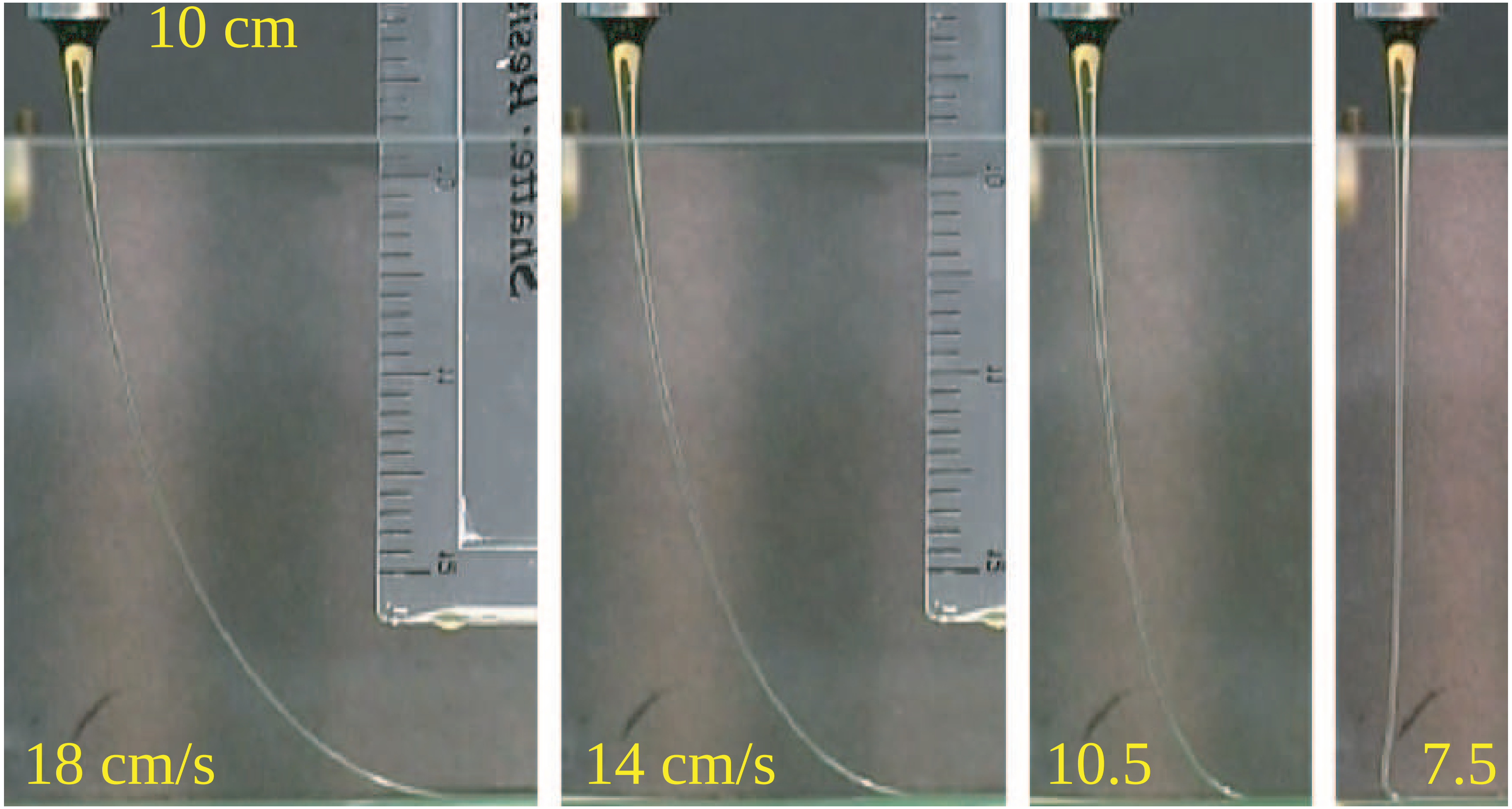
- Belt speed U_b and fall height H are the main control parameters
- Let U_f be the 'free-fall' speed

  — speed if $T = 0$ after vertical fall through H

Steady Stretching Catenary Shapes for $U_b \gtrsim U_f$

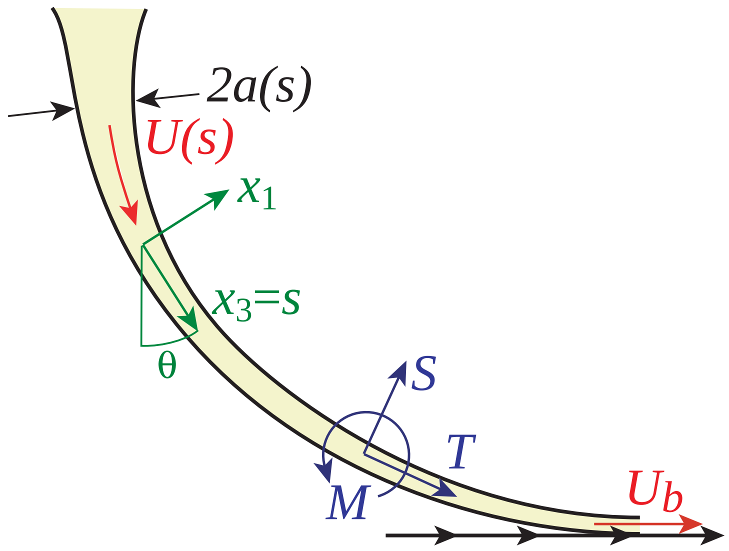
Experiments with Golden Syrup

Chiu-Webster & Lister 2006



Thread is in tension – catenaries with variable mass per unit length

Equations for Steady Catenaries (with bending)



Model by centreline variables

$T = \int \sigma_{33}\, dA \qquad S = \int \sigma_{13}\, dA$

$M = \int x_1 \sigma_{33}\, dA$

Rolling conditions on belt

Clamped conditions at nozzle

Geometry $\theta' = \kappa,\ x' = \sin\theta,\ z' = -\cos\theta$

Mass $a^2 U = const.$

Vertical force balance $(T\cos\theta - S\sin\theta)' = \pi a^2 \rho g$

Horizontal force balance $(T\sin\theta + S\cos\theta)' = 0$

Moment balance $M' = S$

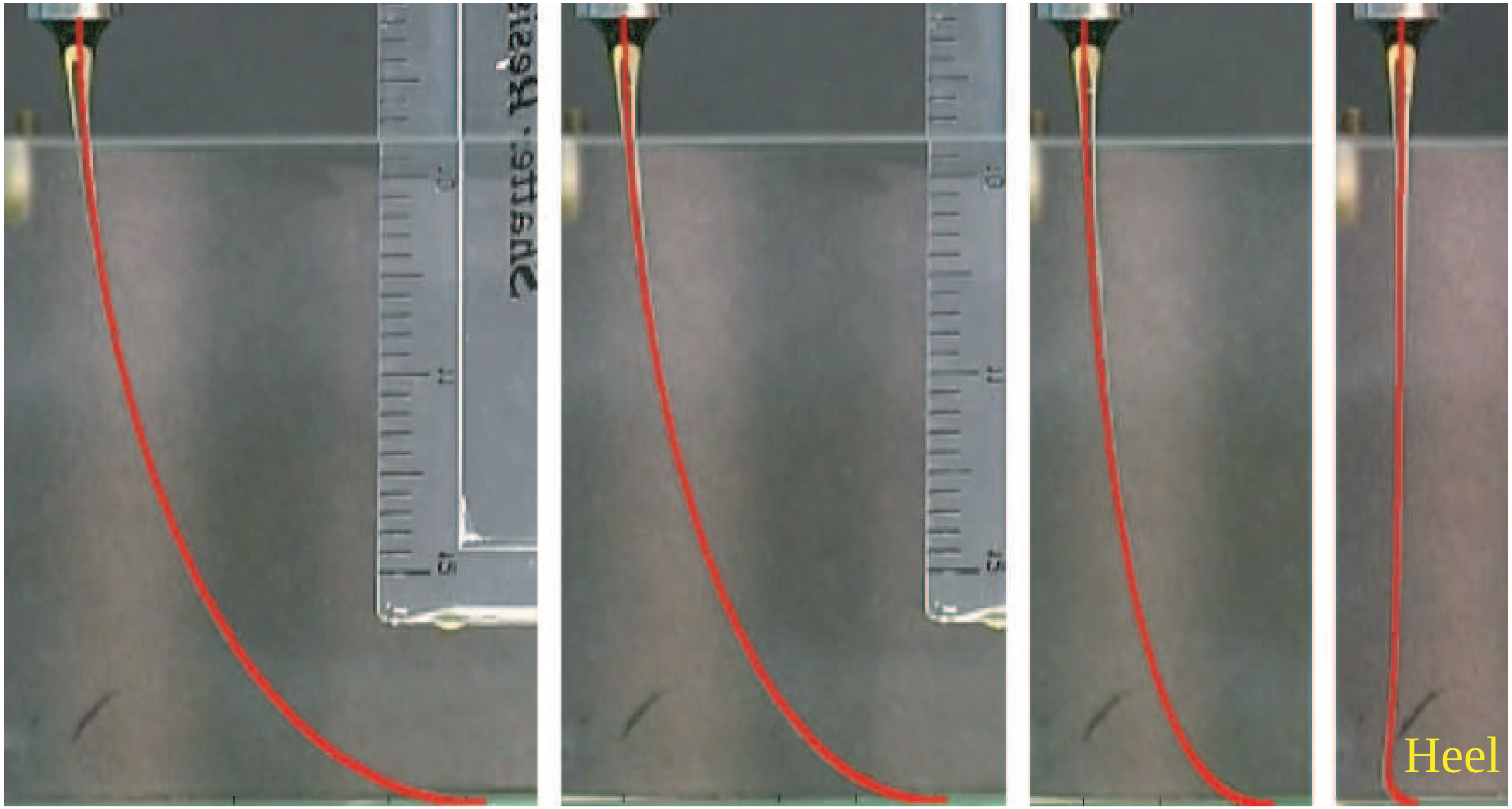
Resistance to stretching $T = 3\mu\pi a^2 U'$

Resistance to bending $M = \frac{3}{4}\mu\pi a^4 U \kappa'$

8th-order ODEs

Steady Stretching Catenary Shapes for $U_b \gtrsim U_f$

Very good agreement between simple theory and experiments



As $U_b \searrow U_f$ the shape develops a heel (to be continued . . .)

Motion in 3D for Coiling Etc. (no inertia for simplicity)

Shape $\mathbf{r}(s,t)$ Tangent $\mathbf{d}_3 = \mathbf{r}'$ and material $\mathbf{d}_1, \mathbf{d}_2$

Kinematics $\dfrac{D\mathbf{r}}{Dt} = \mathbf{U}$, $\dfrac{D\mathbf{d}_i}{Dt} = \boldsymbol{\omega} \wedge \mathbf{d}_i$, $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + U_s \dfrac{\partial}{\partial s}$

Mass conservation $\dfrac{Da^2}{Dt} = -(\mathbf{U}')_3 a^2$

Forces & moments $\mathbf{T}' + \mathbf{F} = \mathbf{0}$, $\mathbf{M}' + \mathbf{d}_3 \wedge \mathbf{T} + \mathbf{G} = \mathbf{0}$

Constitutive eqns $\mathbf{M} = \frac{1}{4}\pi a^4 \mu (3\mathbf{I} - \mathbf{d}_3\mathbf{d}_3) \cdot \boldsymbol{\omega}'$ $T_3 = 3\pi a^2 \mu (\mathbf{U}')_3$

for the two bending directions, twisting and stretching

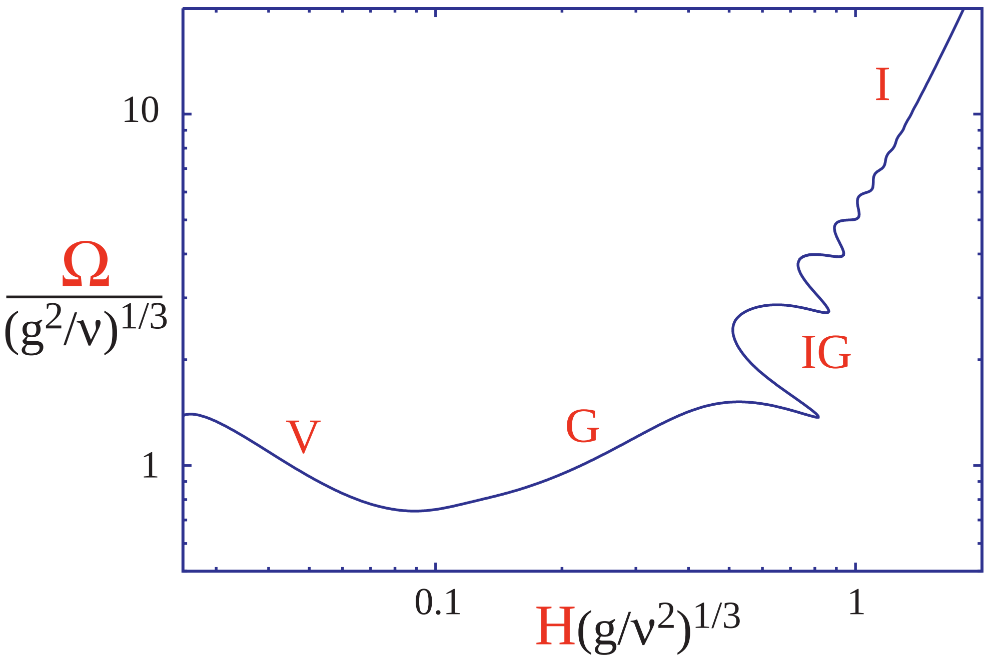
- System for velocities and angular velocities is a 12th order PDE
  The kinematics of position, angle, radius and arclength
  adds another 8 (or 9) orders!

The Four Regimes of Steady Coiling

Ribe et al. 2006

Numerical solutions show four regimes.

Backed by experiments. Explained by scaling arguments.



- Viscous: global buckling (like toothpaste)
- Gravitational: balance gravity and bending stresses in coil
- Inertial: balance inertia and bending stresses in coil
- Inertio-Gravitational: (see next slide)

Oscillations of a Hanging String

Vertical hanging string, uniform radius a, negligible stiffness

Vertical force balance $T' = \pi a^2 \rho g \Rightarrow$ linear variation of tension

Small horizontal displacements $(Tx')' = \pi a^2 \rho \ddot{x}$

Seek eigenmodes $x = X(s)e^{i\Omega t}$

$$\frac{d}{ds}\left(s\frac{dX}{ds}\right) = -\frac{\Omega^2}{g}X, \quad X(H) = 0, \quad sX' \to 0 \text{ at } 0$$

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Solution $X = J_0(2\Omega\sqrt{s/g})$ with eigenfrequencies $\Omega = \dfrac{j_{0,n}}{2}\sqrt{\dfrac{g}{H}}$

Frequencies modified by nonuniform radius

and by Coriolis effects if there is velocity U along string

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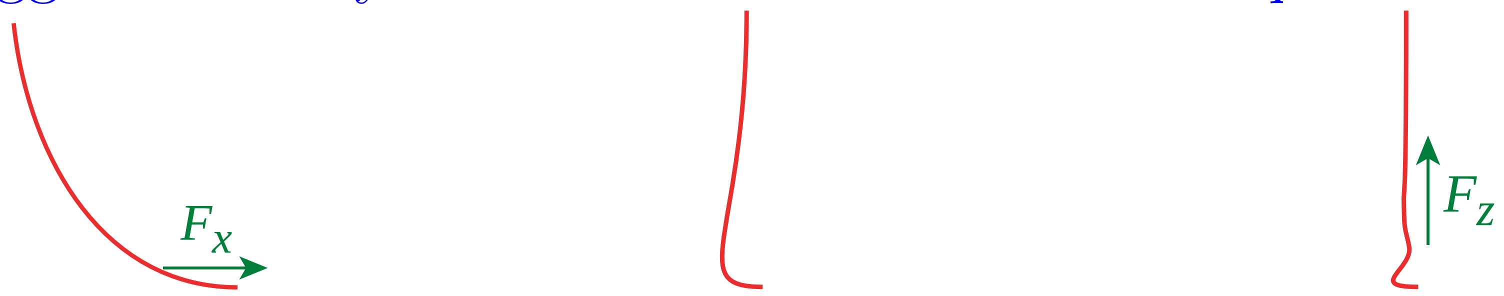
IG regime is due to resonance of coiling with the pendulum modes

Back to the Dragged Thread

- There are three basic regimes (and steady solutions in each):

Dragged catenary | Transitional Heel | Compressional Heel



$U_b > U_f$ | $U_b \approx U_f$ | $U_b < U_f$

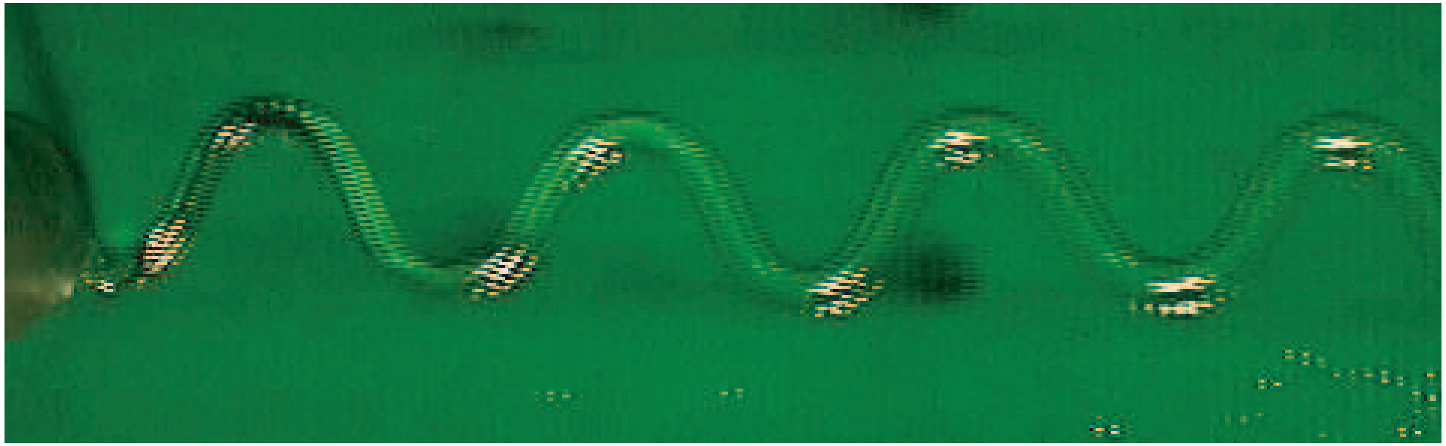
- Expect compressional heel to be unstable to buckling
- Linear perturbations to the planar basic state decouple into

  a 12th order out-of-plane system (bending & twisting)

  a 9th order in-plane system (bending & stetching)
- At leading order, transitional heel becomes unstable not when $F_z = 0$ but when $F_x = 0$. Not buckling but overbalancing!

Onset of Oscillations for $U_b = U_f - O(\epsilon^{1/2})$

Thread becomes unstable to sinusoidal meanders



$H = 8.5\,cm$

$U_b = 4.0\,cm/s$

$\Leftarrow$

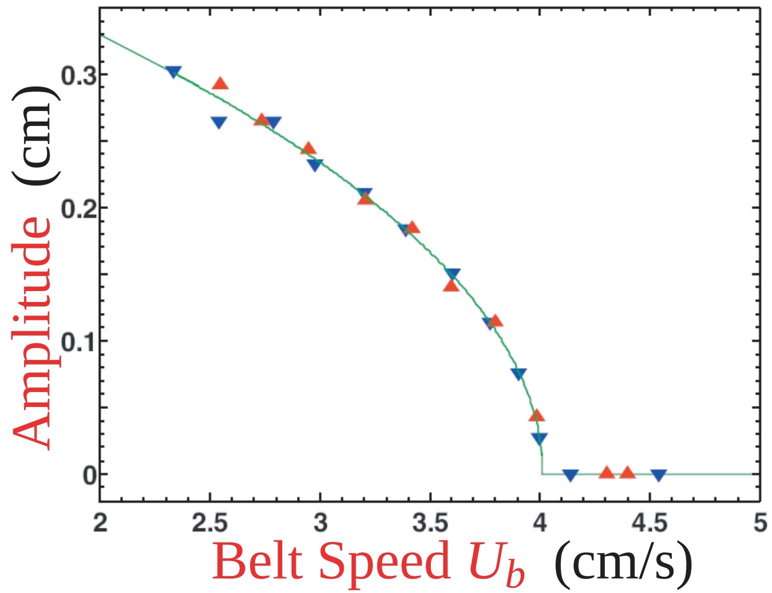
Onset is a Hopf bifurcation

Onset point and frequency can be calculated

numerically (Ribe, Lister & C-W 2006)

asymptotically (Blount & Lister 2011)

e.g. $\Omega \sim \frac{U_f}{H}\frac{2.53}{(\epsilon \ln \epsilon)^{1/2}}$ for $\epsilon = \frac{a_f}{H} \ll 1$



Morris et al. 2008

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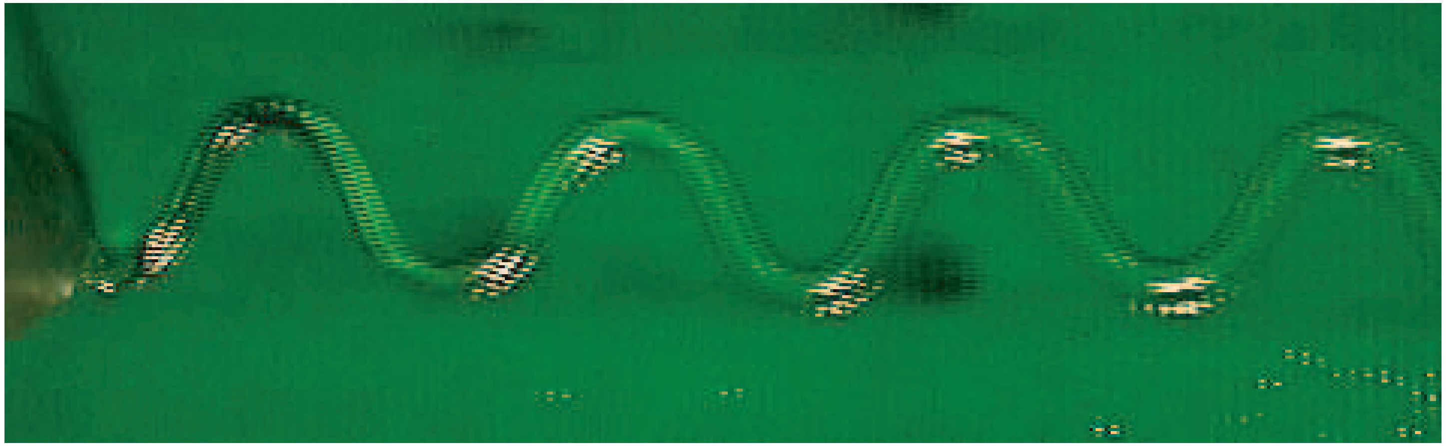

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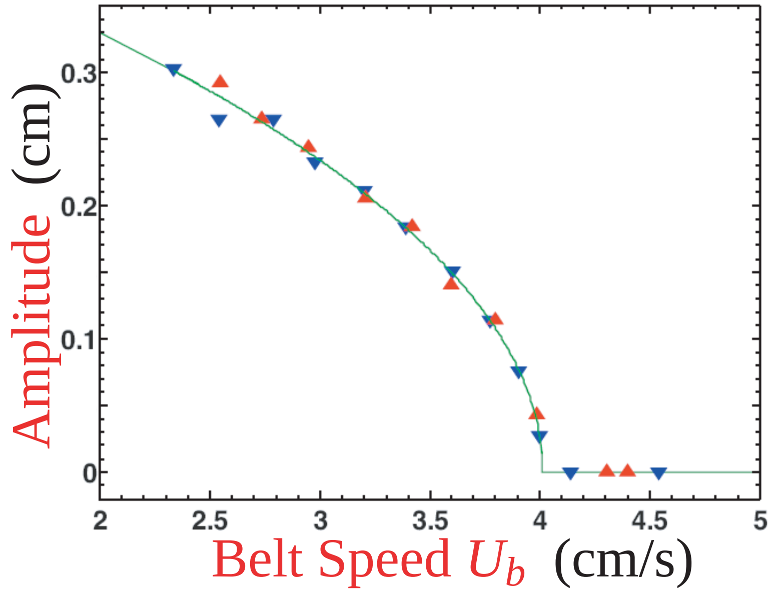
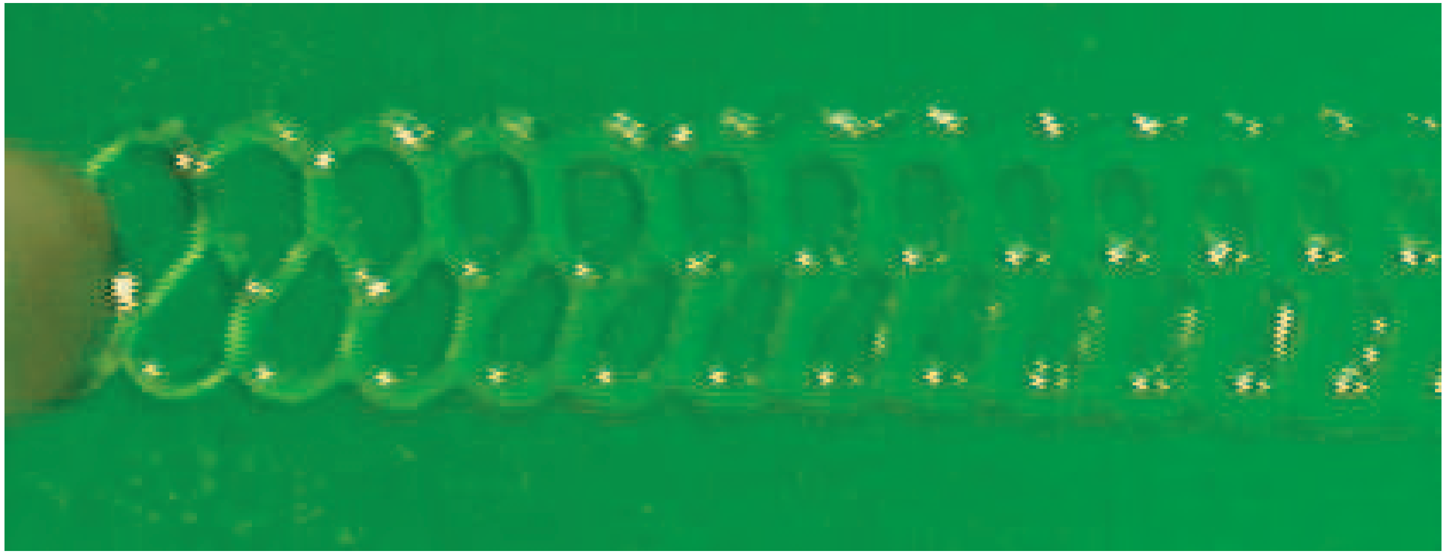


Morris et al. 2008

Further Bifurcations for $U_b \lesssim U_f$

Braiding

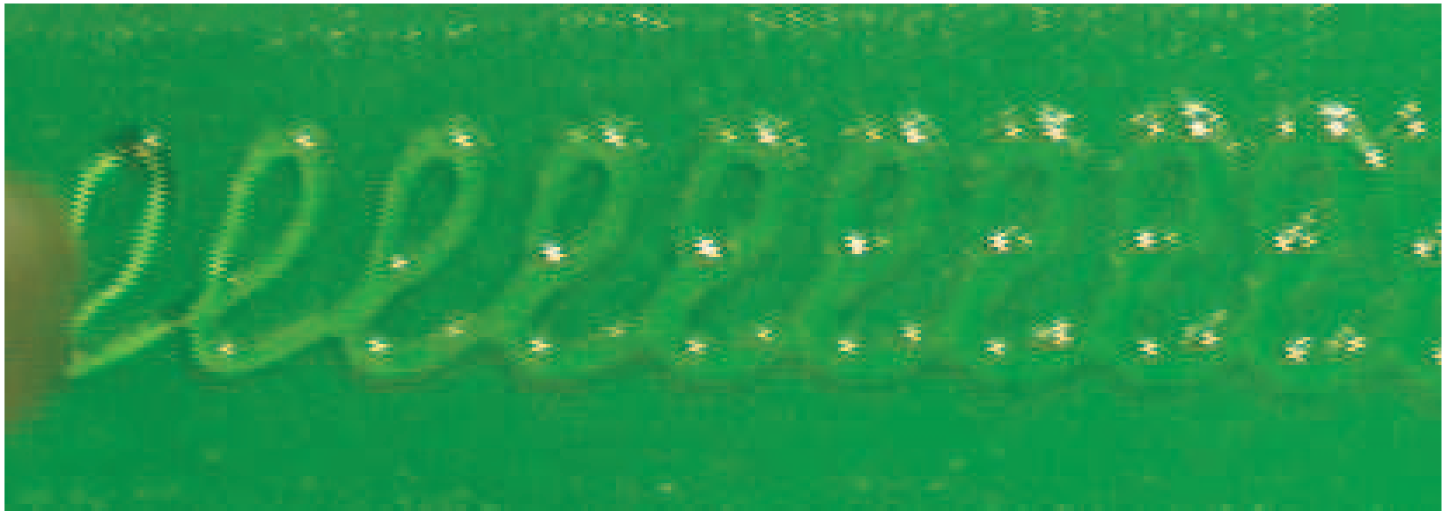


$H = 8.25\,cm$

$U_b = 1.1\,cm/s$

$\Leftarrow$

Slanted loops



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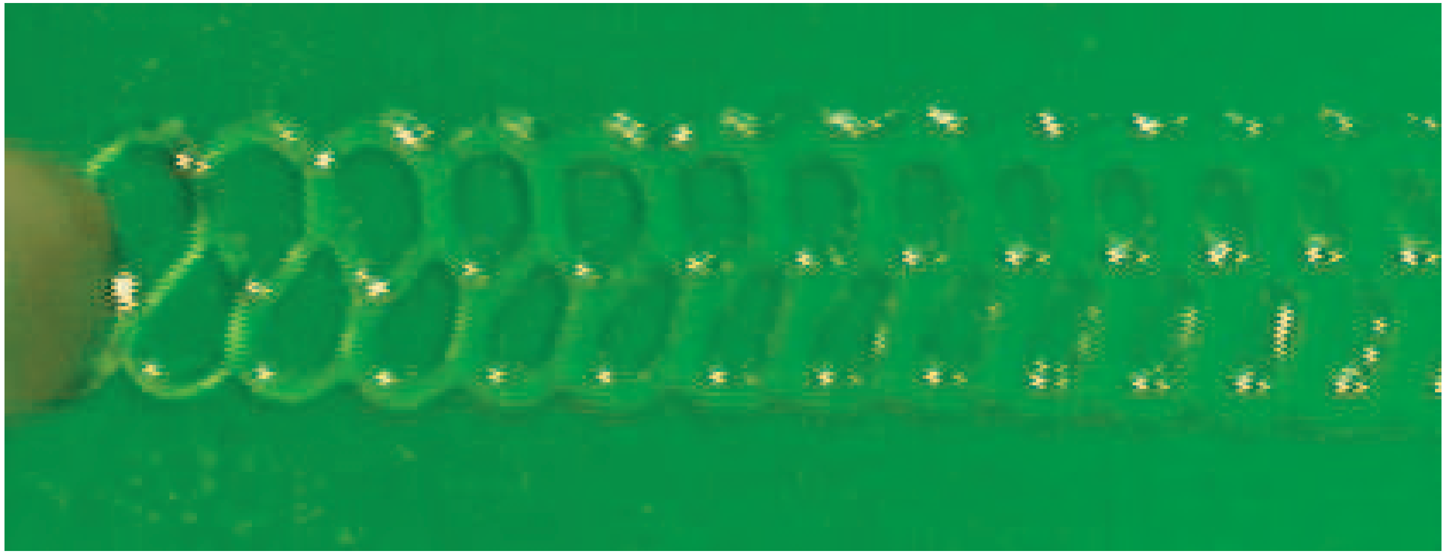

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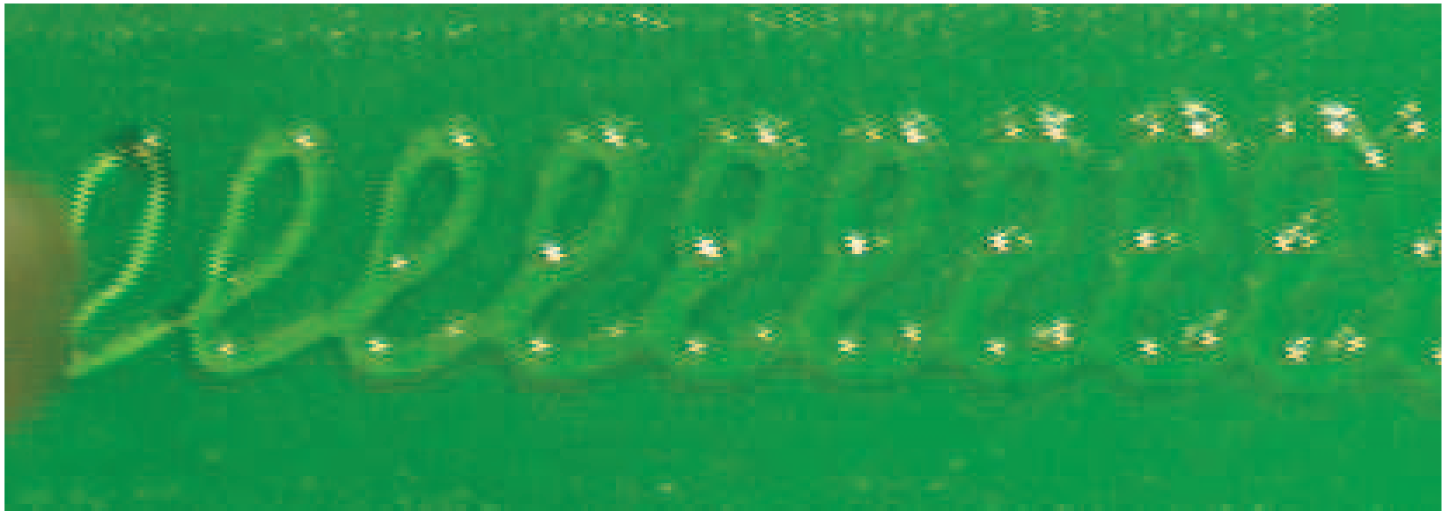

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Coiling – the obvious result as $U_b \to 0$

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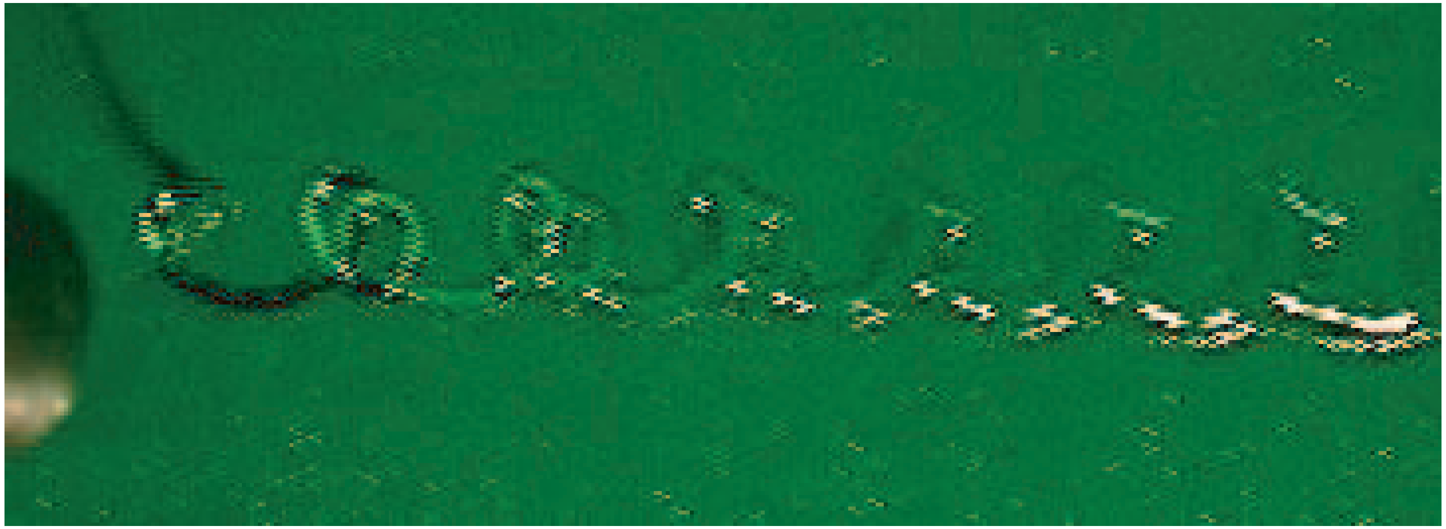
Double coiling



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$\Leftarrow$

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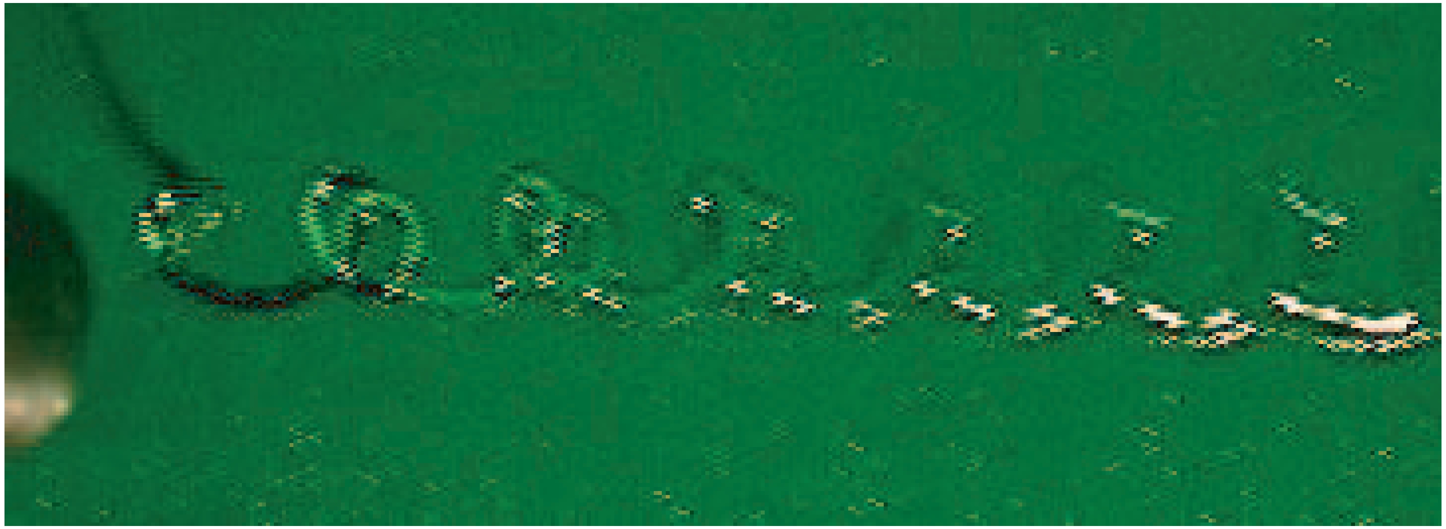

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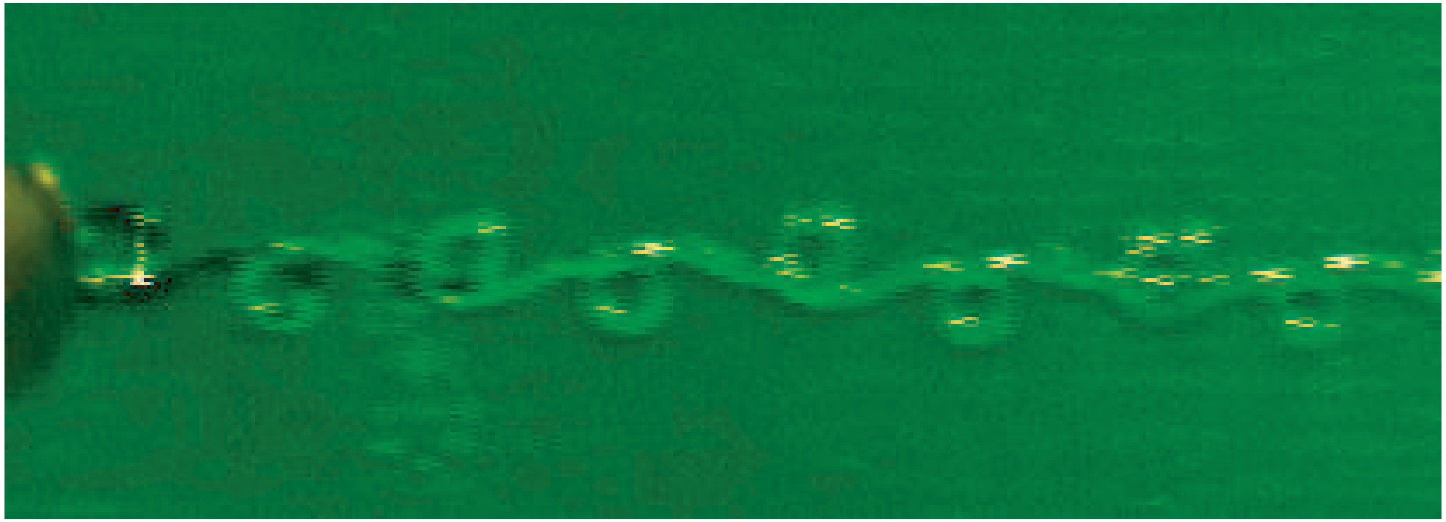
Figure-of-eight



$H = 9.5\,cm$

$U_b = 3.3\,cm/s$

$\Leftarrow$

W

$H = 8.5\,cm$

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$\Leftarrow$

Further Bifurcations for $U_b \lesssim U_f$

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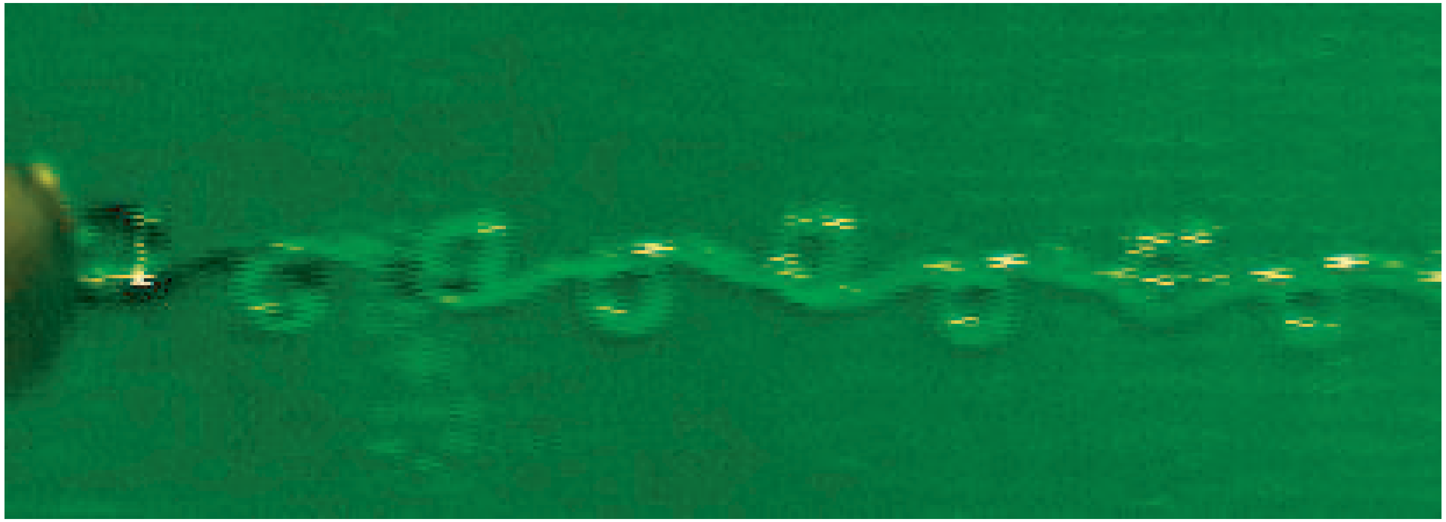

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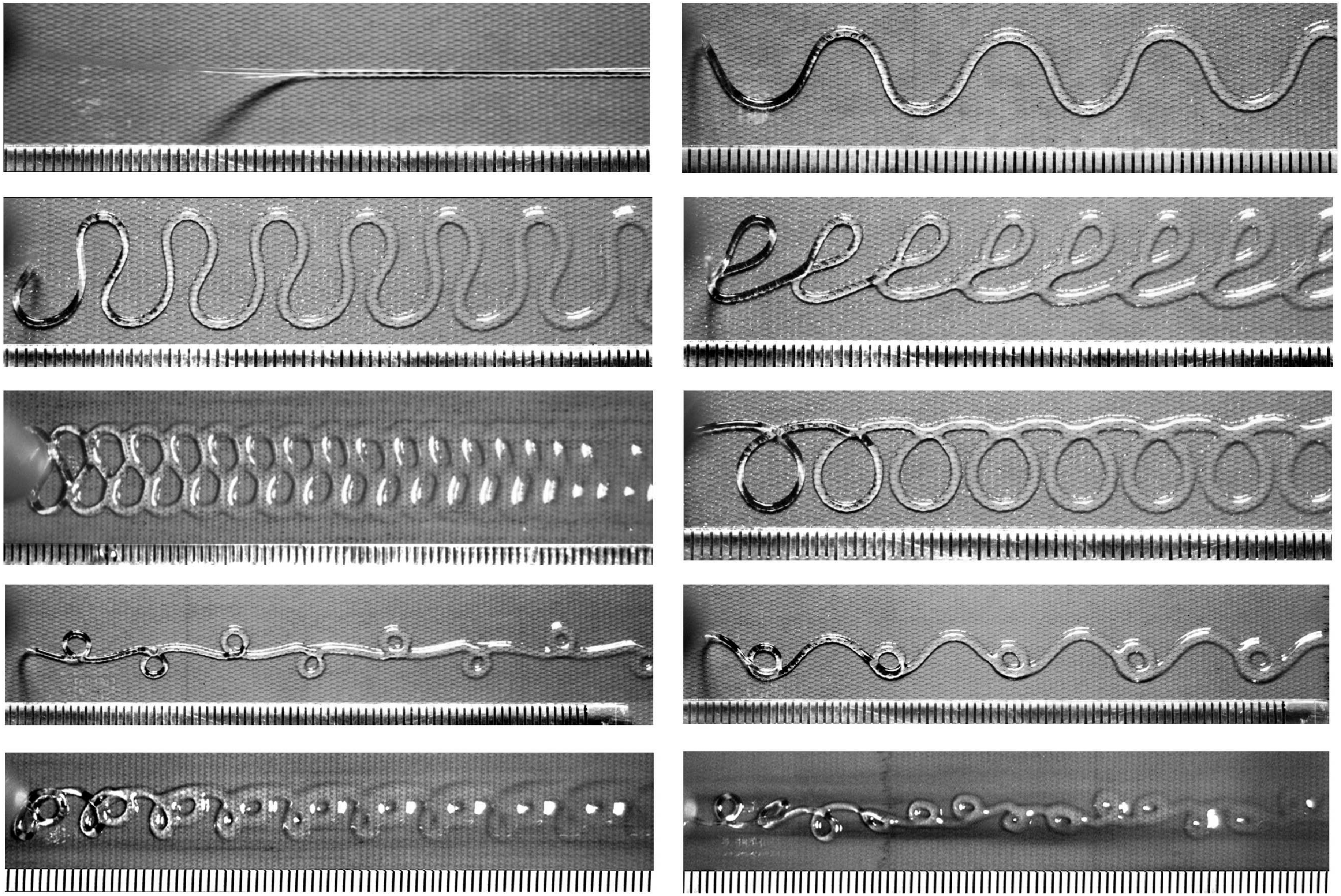
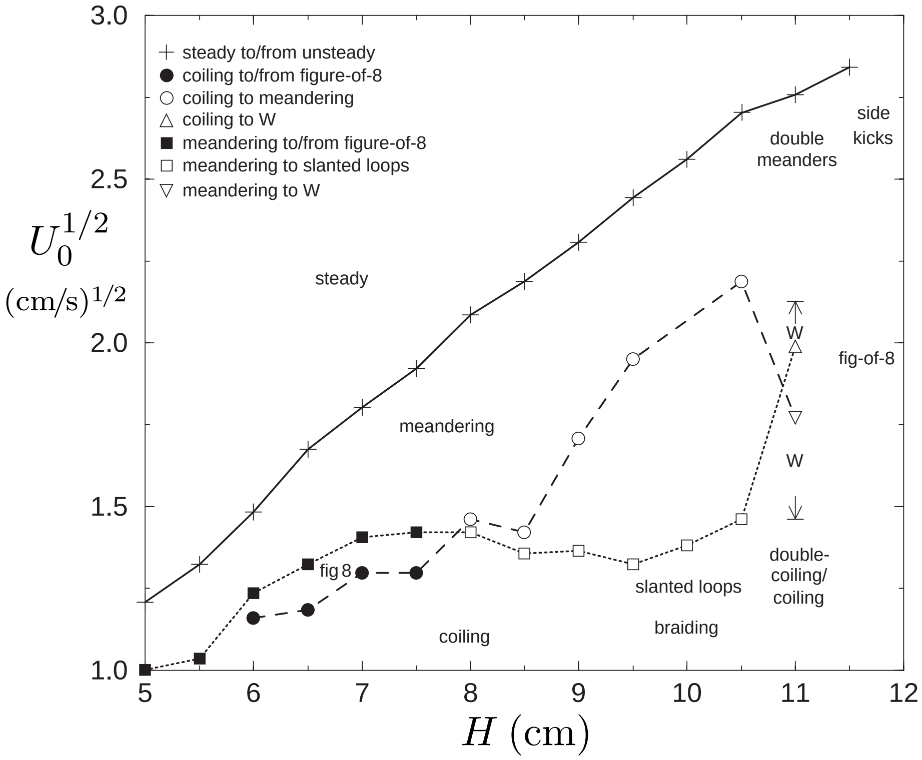
Patterns from a Fluid-Mechanical Sewing Machine

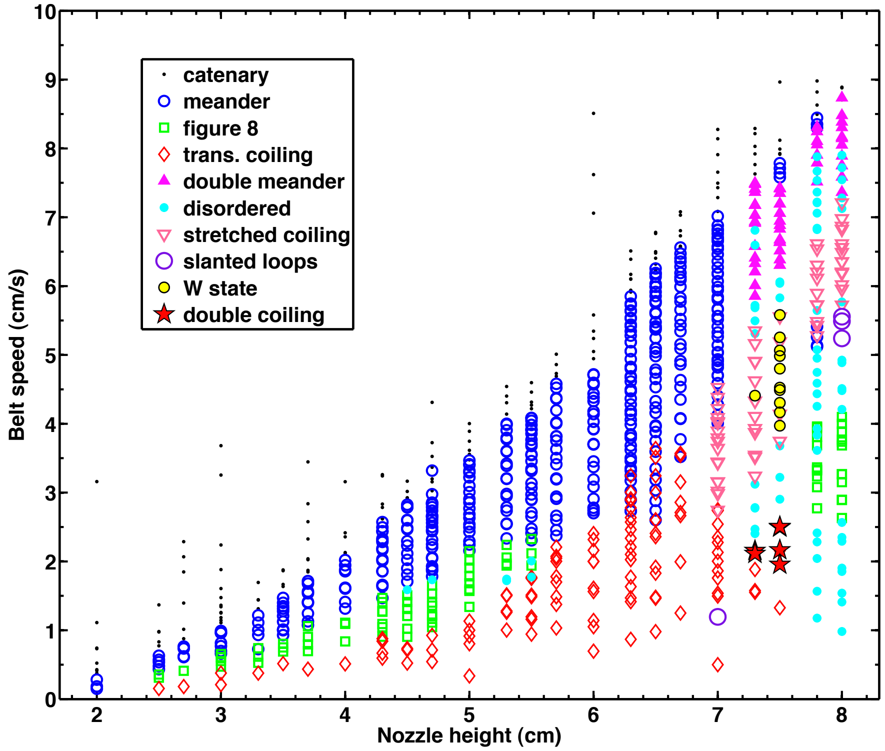


Regime Diagrams

The control space (H, U_b) can be explored systematically



Chiu-Webster & Lister 2006

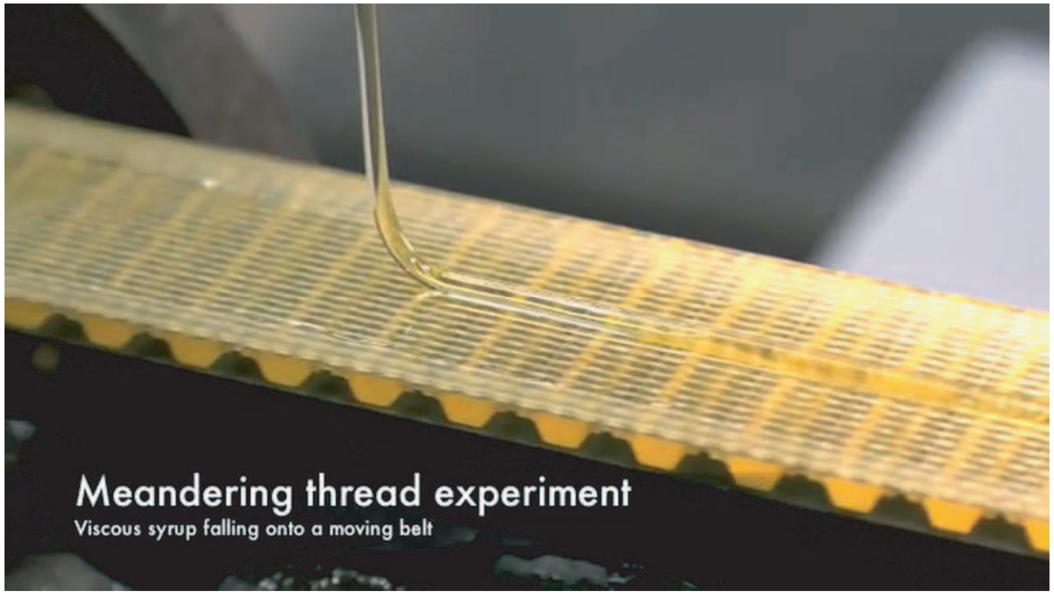


Morris et al. 2008

Origins of Complexity

- Unsteadiness driven by overbalancing of the compressive heel
- Weak restoring by deflection of the tail
- also get resonances with pendulum modes in the tail
- also self-intersection on belt can lock/disrupt patterns

This is a beautiful system to show that the physics of viscous flow can be fun!



This movie by Stephen Morris is on U-Tube. The belt speed decreases slowly, keeping everything else constant.

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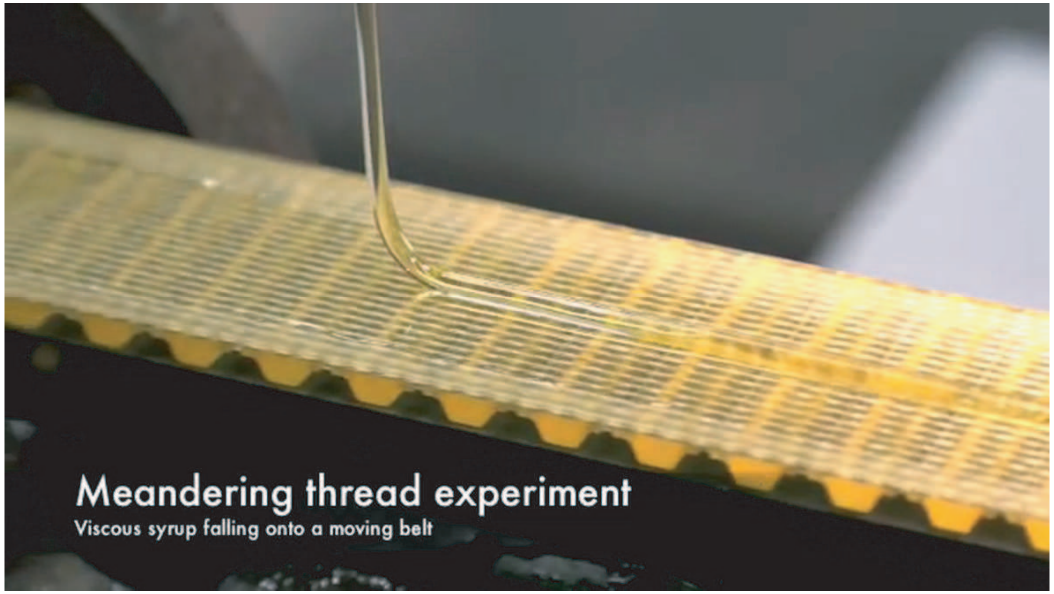

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