# What is Brownian motion?

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12 October 2009

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In 1827, the Scottish botanist Robert Brown observed pollen grains under his microsope.

- The mathematical model of the observed trajectories of the pollen is called Brownian motion.

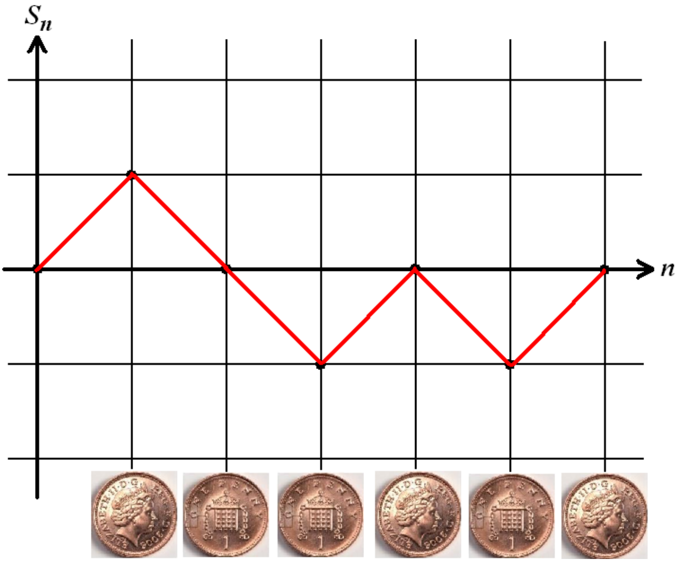
Answer 1:

## Definition

A *Brownian motion* is a continuous stochastic process with independent and stationary increments.

## Random Walk

- Flip a penny repeatedly.
- $$X_n = \begin{cases} 1 \\ -1 \end{cases} \text{if the } n\text{-th flip comes up } \begin{matrix} \text{heads} \\ \text{tails} \end{matrix}$$
- $S_n = X_1 + \ldots + X_n$


$S_n$
$n$

In *discrete time* the increments of a random walk

$$S_{n_1} - S_{n_0}, \ldots, S_{n_k} - S_{n_{k-1}}$$

- are **independent**
- and **stationary**, i.e. the distribution of $S_{m+n} - S_m$ does not depend on $m$

Note

- $\mathbb{E}(S_n) = 0$
- $\mathrm{Var}(S_n) = n$

so the central limit theorem says $S_n/\sqrt{n}$ is approximately Gaussian:

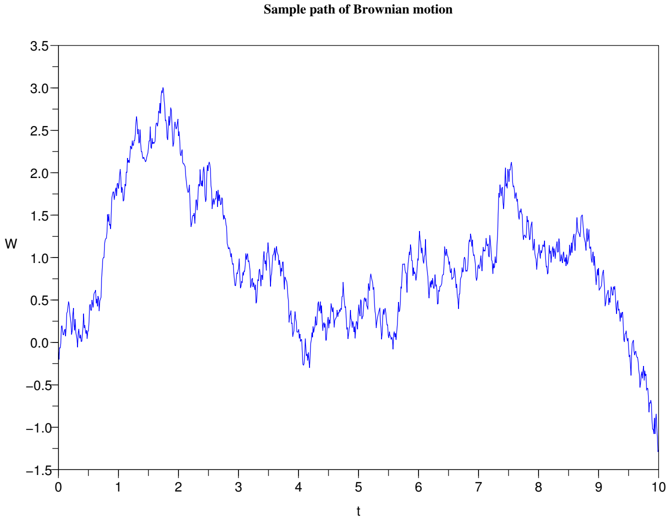
$$\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \leq x\right) \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

as $n \to \infty$.

Let

$$W_t^{(n)} = \frac{S_{nt}}{\sqrt{n}}$$

- It turns out that $(W_t^{(n)})_{t \geq 0}$ converges as $n \to \infty$.
- The limit $W = (W_t)_{t \geq 0}$ is a (standard) Brownian motion.


Sample path of Brownian motion
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
W
0
1
2
3
4
5
6
7
8
9
10
t

## The existence theorem

### Theorem (Wiener)

*Brownian motion exists.*

A *probability space* is

- a set $\Omega$ of possible outcomes of an experiment,
- a collection of subsets (events) of $\Omega$, and
- a function $\mathbb{P}$ that assigns to each event $A$ a probability $\mathbb{P}(A) \in [0, 1]$.

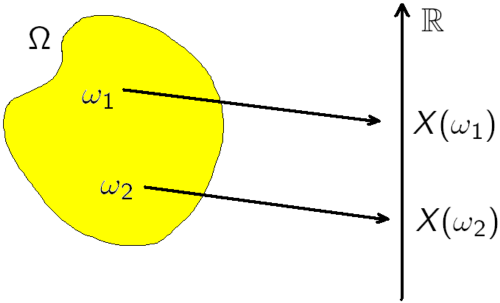
The function $\mathbb{P}$ must be additive: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ when $A$ and $B$ are disjoint events.

**Example.** Flip a penny twice. The possible outcomes are

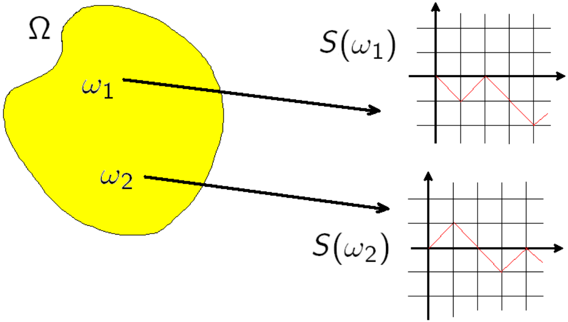
Each outcome is equally likely, so

$$\mathbb{P}(\text{at least one head}) = \mathbb{P}(HH, HT, TH) = \frac{3}{4}$$

A *random variable* $X$ is a function that assigns each outcome $\omega \in \Omega$ a real number $X(\omega)$.



A *stochastic process* $S$ is a function that assigns each outcome $\omega \in \Omega$ a function $t \mapsto S_t(\omega)$.



## Gaussian processes

- A stochastic process $(X_t)_{t\geq 0}$ is Gaussian if

$$X_{t_1}, \ldots, X_{t_n}$$

  are jointly normal.
- The increments $W_{u+t} - W_u$ of Brownian motion are normal with mean 0 and variance $t$.

Answer 2:

### Theorem

*Brownian motion is the unique continuous Gaussian process $W$ with*

- $\mathbb{E}(W_t) = 0$
- $\mathbb{E}(W_s W_t) = \min\{s, t\}$

Answer 3:

### Theorem (Karhunen–Loeve)

*Let $\xi_1, \xi_2, \ldots$ be independent $N(0,1)$ random variables, and let*

$$W_t = \sum_{n=1}^{\infty} \frac{2\sqrt{2}}{(2n-1)\pi} \sin\left(\frac{(2n-1)t\pi}{2}\right) \xi_n.$$

*Then $(W_t)_{t\in[0,1]}$ is Brownian motion.*

### Theorem

*With probability one, the sample paths of Brownian motion are nowhere differentiable.*

Theorem

*If $(W_t)_{t \geq 0}$ is a Brownian motion, so are*

- $\frac{1}{c} W_{c^2 t}$,
- $W_{t+t_0} - W_{t_0}$, *for fixed* $t_0$, *and*
- $t W_{1/t}$

**Markov processes**

Brownian motion is a Markov process:

$$\mathbb{E}[f(W_T)|W_{t_0} = w_0, \ldots, W_{t_n} = w_n] = \mathbb{E}[f(W_T)|W_{t_n} = w_n]$$

for all $0 \leq t_0 \leq \ldots \leq t_n \leq T$.

Let $X = (X_t)_{t \geq 0}$ be a time-homogeneous Markov process.

- Let $P_t$ be the operator such that

$$(P_t f)(x) = \mathbb{E}[f(X_t) | X_0 = x].$$

- The operator

$$Q = \lim_{t \downarrow 0} \frac{P_t - I}{t}$$

is called the generator (or $Q$-matrix) of $X$.

Answer 4:

### Theorem

*Brownian motion is the unique continuous Markov process with generator*

$$Q = \frac{1}{2}\frac{\partial^2}{\partial x^2}.$$

Let $W$ be Brownian motion.

**Theorem (Heat equation)**

$$V(t,x) = \mathbb{E}[f(W_t)|W_0 = x]$$

*if and only if*

$$\frac{\partial V}{\partial t} = \frac{1}{2}\frac{\partial^2 V}{\partial x^2}, \quad V(0,x) = f(x)$$

Let $D \subset \mathbb{R}^n$ and $W$ be $n$-dimensional Brownian motion.

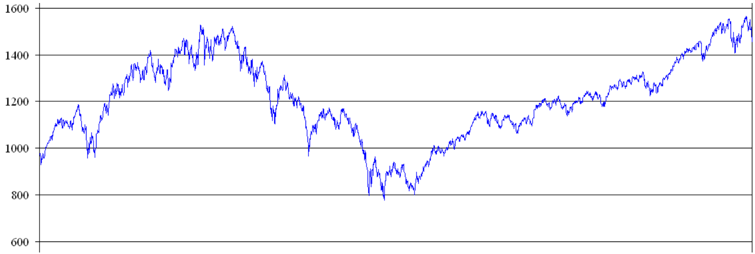
Theorem (Laplace equation)

$$U(x) = \mathbb{E}[f(W_\tau)|W_0 = x], \quad \tau = \inf\{t \geq 0 : W_t \in \partial D\}$$

*if and only if*

$$\Delta U = 0 \text{ in } D, \quad U = f \text{ in } \partial D$$

## Black–Scholes PDE



Black and Scholes modelled the price of a stock by

$$S_t = e^{\mu t + \sigma W_t}.$$

Under some conditions, they showed the price of a call $C$ of a call option satisfies

$$\frac{\partial C}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} = rC$$

with terminal condition $C(T, S) = (S - K)^+$ where $r$ is the risk-free rate.

**Complex Brownian motion**

Let $B_t = W_t + i\tilde{W}_t$, where $W$ and $\tilde{W}$ are independent Brownian motions, and $i = \sqrt{-1}$.

### Theorem

*Let $f$ be a holomorphic. Then there exist a complex Brownian motion $Z$ and a non-decreasing process $A$ such that*

$$f(B_t) = f(0) + Z_{A_t}.$$

### Corollary

$\mathbb{P}(B_t = \alpha$ *for some* $t > 0) = 0$ *for all* $\alpha \in \mathbb{C}, \alpha \neq 0$.

### Corollary (Liouville)

*If f is bounded, then f is constant.*

## Martingales

A *martingale* $M$ is a stochastic process such that

$$\mathbb{E}(M_T | M_{t_0} = m_0, \ldots, M_{t_n} = m_n) = m_n$$

for $0 \leq t_0 \leq \ldots \leq t_n \leq T$.

Answer 5:

### Theorem (Lévy)

*Brownian motion is the unique continuous martingale $W$ such that $W_t^2 - t$ is also a martingale.*

A martingale is symmetric if the conditional distribution of $M_t - M_s$ is the same as that of $M_s - M_t$.
Answer 6:

### Theorem (MT)

*Brownian motion is the unique continuous symmetric martingale $W$ such that*

$$W_t \sim N(0, t).$$