The Uses of Infinity—Emergence and Reduction Reconciled?

Is the whole greater than the sum of the parts? Holism or reduction?

1. The jargon

Emergence as behaviour that is both novel and robust, especially in comparison to a theory of the microscopic details. (So emergence is not just 'good' variables and-or approximation schemes.)

Reduction: The core idea is that one theory is a *definitional extension* of another.

> Theory $T_b = T_{\text{bottom/basic/best}}$ reduces $T_t = T_{\text{top/tangible/tainted}}$; or T_t reduces to, is reducible to, T_b

when for all words used by T_t, there are definitions in the language of T_b such that when all the definitions are added to T_b, you can derive all of T_t. (The definitions may be very hard to find; and very long.)

Example?: Thermodynamics $= T_t$ uses 'temperature', 'pressure' etc.; statistical mechanics $= T_b$ uses 'position', 'momentum', 'molecule', 'probability'.

Supervenience: Total matching of any two objects as regards one family of properties (called the *subvening* family, say $\mathcal{B}$) implies their total matching as regards the other family (the *supervening* family, say $\mathcal{T}$).

Example?: The mental supervenes on the physical: if a cat sees yellow, an atom-for-atom replica of the cat also sees yellow.

Supervenience is a weakening of definitional extension: it allows one or more of the definitions (of a property $T \in \mathcal{T}$ in terms of the various $B \in \mathcal{B}$) to be infinitely long: an infinite disjunction (...or...or...or......) of "ways to be T".

2. Two claims

(1) Emergence is compatible with definitional extension.

I will give three examples, each with a parameter $N = \infty$; (N is the number of 'degrees of freedom').

And for each: choosing a salient weaker theory using finite N blocks the reduction.

(2) Supervenience is a red herring. It is scientifically useless because it gives no control on the infinite disjunction; in particular, no kind of limit is taken.

Compare continuous models of fluids (e.g. sound, flow), which model a finite system, i.e a system with finitely many degrees of freedom (atomism!), as infinite.

Three obvious justifications of $N = \infty$, which are shared with such models of fluids.

1: Mathematical convenience: impossible to overstate!

2: Elimination of finitary effects.

3: Empirical success: the proof of the pudding is in the eating.

So, agreed: $\infty - 10^{23} = \infty$ and N is actually finite! But if $f(10^{23}) \approx f(10^{46})$ etc, then it is good, and sometimes even indispensible, to model the system with $f(\infty)$.

Agreed: maybe reduction needs more than just definitional extension; maybe the defined properties have to be already in, or even central to, the reducing theory. But in our examples, T_b rich enough to consider $N = \infty$ *does* contain the defined properties.

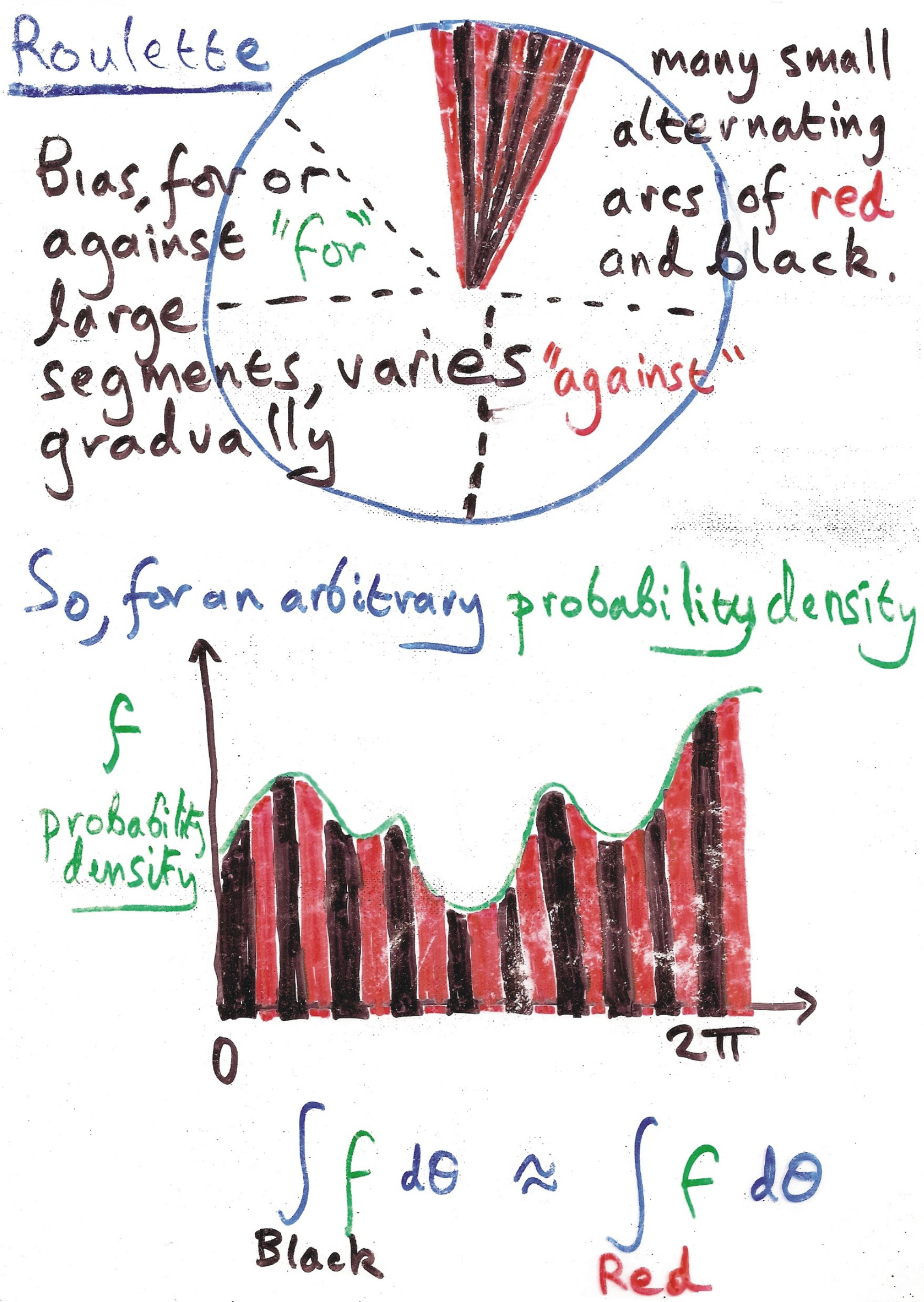

Figure 1: The method of arbitrary functions

3. The method of arbitrary functions

A roulette wheel, with unknown biassing in spin and friction. But suppose we assume:

(i): there are a *large* number N of alternating arcs of red and black;

(ii): the biassing favours and disfavours *big* segments;

(iii): within a single big segment, the bias is "smooth": adjacent arcs get a similar bias.

Then we can be confident that each long-run frequency is about 50%. For any biassing regime, no matter how wiggly (sensitive to angular position), can be washed out so as to give equiprobability, by considering a sufficiently large N.

Indeed:

For any $M \in \mathbb{R}$, for all probability density functions f with derivative bounded by M, $| f' |< M$: as $N =$ the number of arcs, goes to infinity:
$\int_R f \, d\mu \equiv$ prob(Red) $\rightarrow \frac{1}{2}$.

Emergent probabilities
The equiprobability is robust, i.e. holds for many different density functions. And T_t is a definitional extension of T_b, if we take T_b to be a rich enough model of the wheel to include both

(i) the postulation of various possible density functions f; and

(ii) consideration of the infinite limit $N = \infty$.

The emergent behaviour, i.e. equiprobability, is frustrated if we confine ourselves to finitary T_b.

4. Fractals

Fractals form a recent episode in a grand narrative across 400 years of mathematics, the generalization of the idea of a mathematical function: Galileo, Euler—and Coverly.

Philosophy is written in this immense book that stands ever open before our eyes (I speak of the Universe), but it cannot be read if one does not first learn the language and recognize the characters in which it is written. It is written in mathematical language, and the characters are triangles, circles and other geometrical figures, without the means of which it is humanly impossible to understand a word; without these philosophy is a confused wandering in a dark labyrinth. (Galileo, *Il Saggiatore* 1623, *Opere VI*, 197.)

d'Alembert vs. Euler, on vibrating strings: d'Alembert (1747) describes the displacement $f(x,t)$ of a vibrating string by:

$$\frac{\partial^2 f}{\partial t^2} = a^2 \, \frac{\partial^2 f}{\partial x^2} \ .$$

Can this describe a plucked string, with a "corner"? d'Alembert says: No.

Euler replies (1748): Yes. The analysis should be generalised 'so that the initial shape of the string can be set arbitrarily ... either regular and contained in a certain equation, or irregular and mechanical'.

Truesdell: 'Euler's refutation of Leibniz [i.e.: Leibniz's claim that natural phenomena can all be described by what we now call analytic functions] was the greatest advance in scientific methodology in the entire century.'

Thomasina: ... Each week I plot your equations dot for dot, and they draw themselves as commonplace geometry, as if the world of forms were nothing but arcs and angles. God's truth, Septimus, if there is an equation for a bell, then there must be an equation for a bluebell: and why not a rose? ... Do we believe nature is written in numbers?

Septimus: We do.

Thomasina: Then why do your equations only describe the shapes of manufacture? ... Armed thus, God could only make a cabinet.

Septimus: He has mastery of equations which lead into infinities where we cannot follow.

Thomasina: What a faint heart! We must work outward from the middle of the maze. We will start with something simple. I will plot this leaf and deduce its equation. You will be famous for being my tutor when Lord Byron is dead and forgotten.

We focus on the idea that a set of spatial points can have a dimension that is *not* an integer. For us: *scaling dimension.*

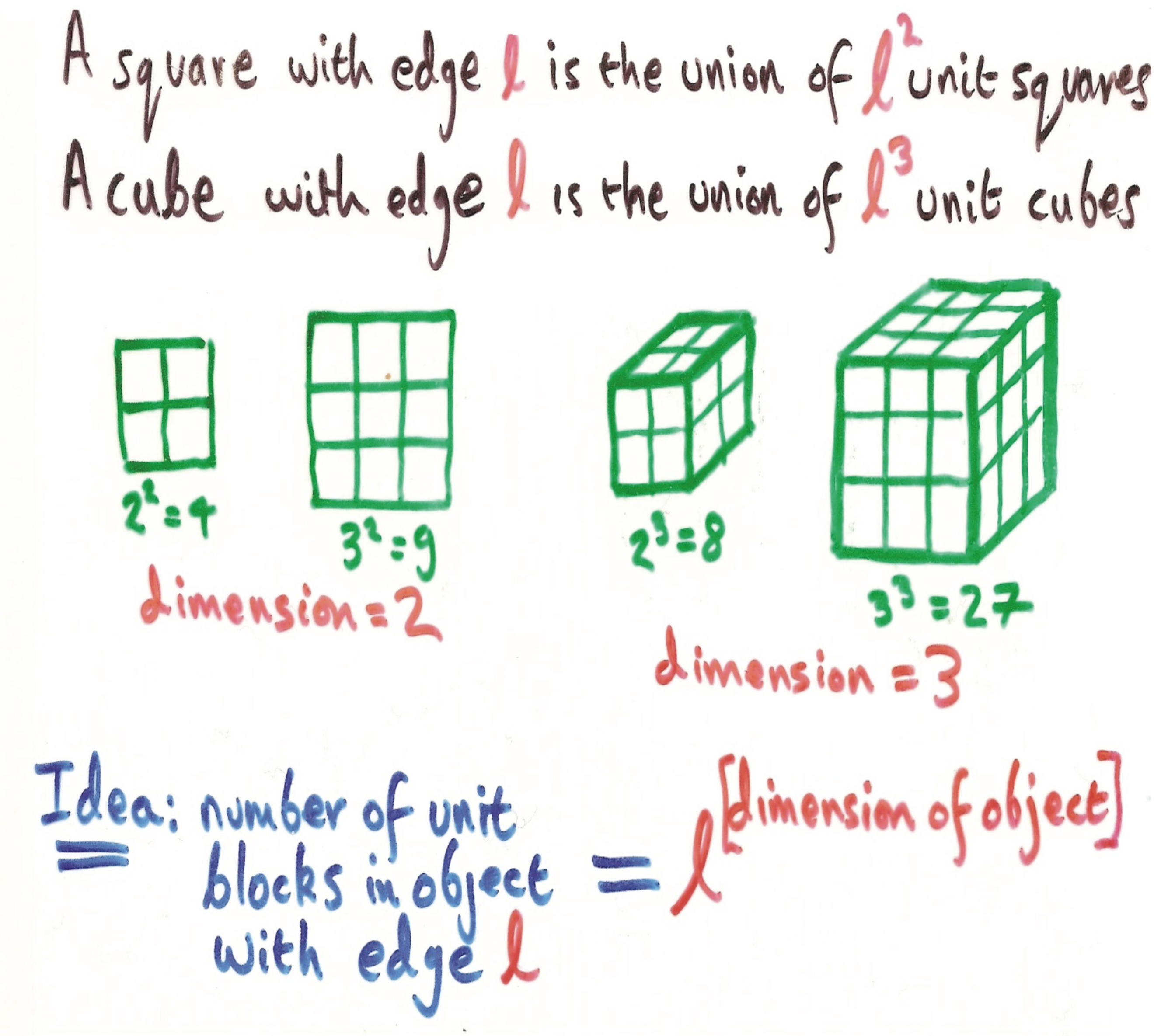

Figure 2: Dimension as an exponent

A square with edge l is the union of l^2 unit squares. For example, a square whose edge is $l = 2$ units long is the union of $2^2 = 4$ unit squares.

A cube with edge l is the union of l^3 unit squares: a cube whose edge is $l = 2$ units long is the union of $2^3 = 8$ unit cubes. Indeed:

$$\text{no. units in object with edge } l = l^{\text{dim of object}}.$$

Applying this to more "pathological" objects, we find that they have non-integer dimensions.

Objects like the Cantor set (1872) *are* self-similar: they *are* unions of shrunken copies of *themselves*—just as much as a square or a cube is...

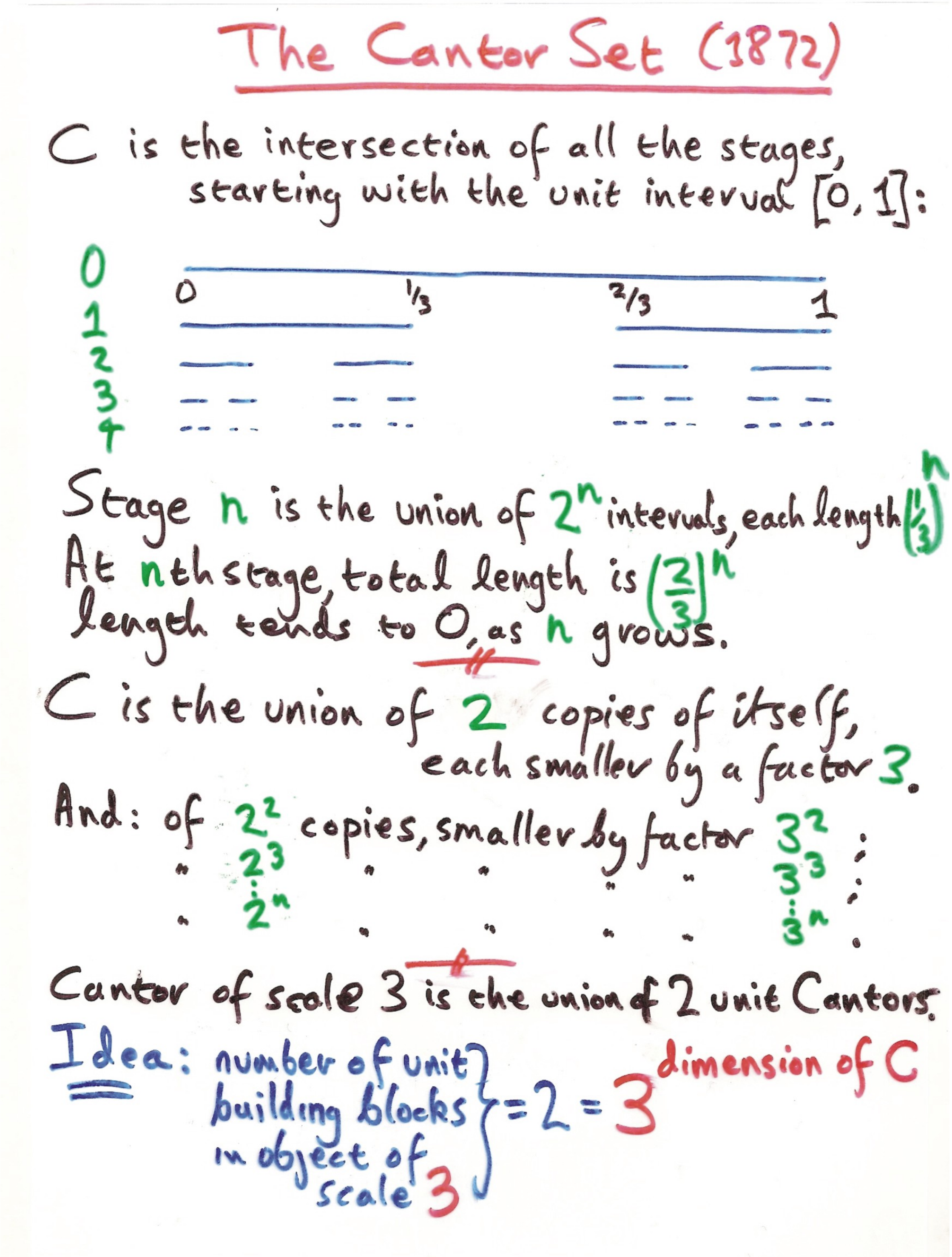

Figure 3: The Cantor set

So what power of 3 is equal to 2? At this point, we recall some elementary logarithms! The idea is that $10^2 = 100$ is re-written as $\log_{10} 100 = 2$. This implies that $2 = 3^{(\log 2/\log 3)}$. So the dimension of C is $\log 2/\log 3$: which is about 0.63.

Similarly for the *Koch snowflake*. Each "side" is the union of four smaller similar curves, each smaller by a factor 3. So applying again the idea of dimension as an exponent, we get:

$$4 = 3^{\text{dimension of Koch}}.$$

So we get:

$$\text{dimension of Koch} = \frac{\log 4}{\log 3} \approx 1.26.$$

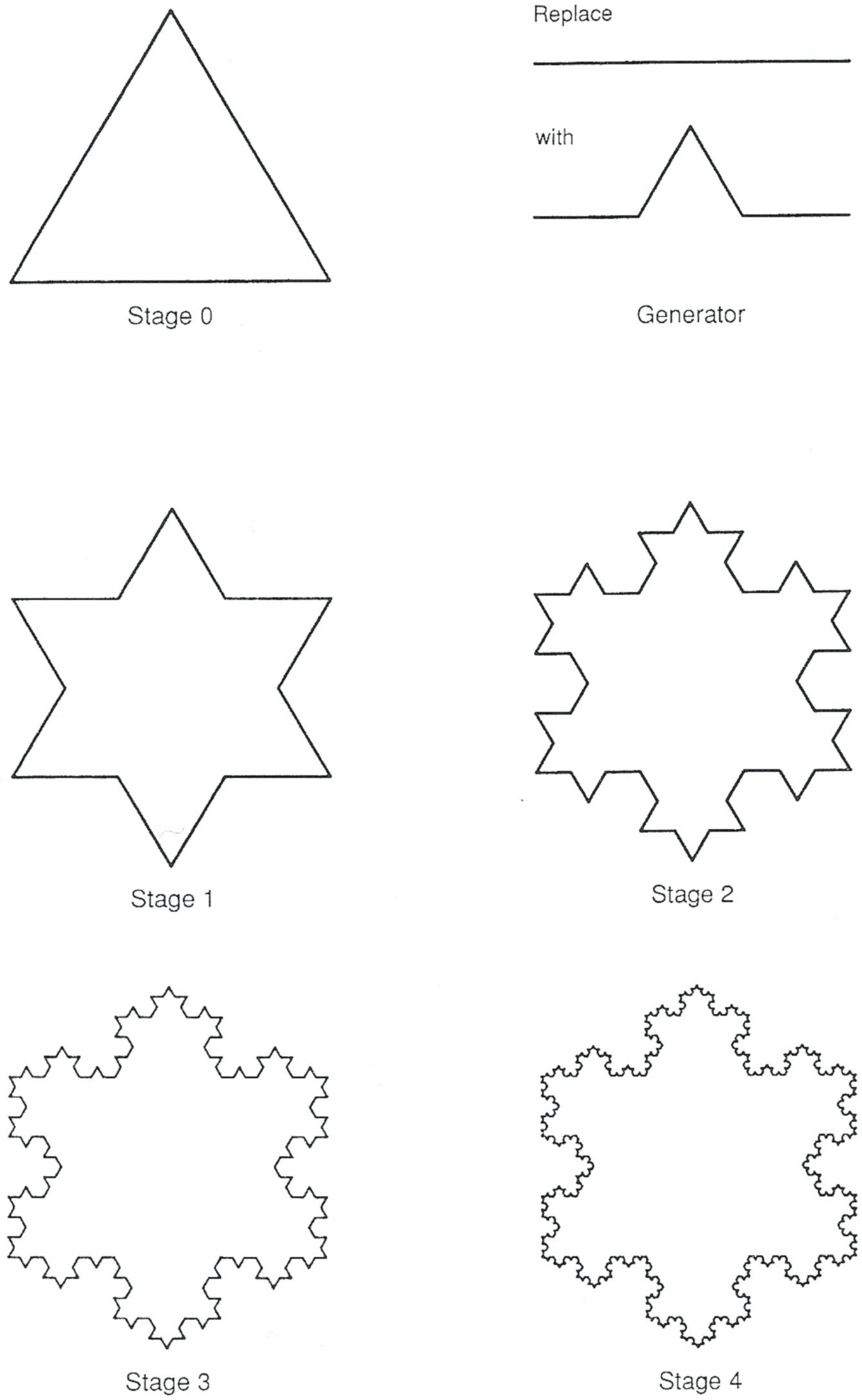

Figure 4: The Koch snowflake

Emergent dimensions

Non-integer dimensions are novel. And they are ‘robust’ in various senses. So indeed: we have emergent dimensions.

Take as T_b: the rich modern theory of scaling dimension (and its cousin notions); and as T_t: the assignment of non-integral dimensions to objects like the Cantor set.

Clearly, T_b contains T_t! But if T_b is just the (salient) traditional theory of dimension, there is no reduction.

Is Fractal Geometry the Geometry of Nature?

Distinguish two questions. First: *do fractals describe the geometry of naturally occurring ("natural history") objects?*

Agreed: It looks like it! Thus in *Arcadia*:

Valentine: If you knew the algorithm and fed it back say ten thousand times, each time there'd be a dot somewhere on the screen. You'd never know where to expect the next dot. But gradually you'd start to see this shape, because every dot would be inside the shape of this leaf. It wouldn't be a leaf, it would be a mathematical object. But yes ...

Figure 5: A fern growing dot by dot

But a leaf and its ilk have no tower of structure, on ever-smaller length-scales. So: 'No'.

Agreed: (i) Often a property obeys a power law, with a resolution raised to a non-integer power; i.e. the exponent is not an integer—as occurs for dimension, with fractals.

(ii): It is heuristically better to have the suggestive language and results of fractal geometry, than just the power law.

Indeed, ever since Lagrange introduced configuration space (1788), physical theories have made indispensable use of various spaces equipped with structures that surely deserve the name 'geometry'. So to the second question—*Do some of our best physical theories use fractals to describe certain subsets of their abstract spaces?*—the answer is: *Yes*.

Example: Statistical mechanics describes aspects of some processes with scale-free (regimes of) theories, involving power-law behaviour on all scales, and fractals.

5. Phase transitions

Statistical mechanics follows thermodynamics in representing phase transitions (like boiling and freezing) by non-analyticities, "discontinuities" in a function (the free energy F).

But these cannot occur for a finite system. So the thermodynamic limit is taken: $N :=$ number of particles, $V :=$ volume $\to \infty$ with $\rho = N/V$ fixed.

This infinite limit brings new mathematical structure. Again, we have three obvious justification: mathematical convenience, elimination of finitary effects, and empirical success.

But what exactly should we say about our (finite!) kettle?

I endorse Mainwood's proposal: for systems with a well-defined thermodynamic limit, $F_N \to F_\infty$, we say: phase transitions occur in the finite system iff F_∞ has non-analyticities.

And if we wish, we can add: *and* if N is large enough, or the gradient of F_N is steep enough. This vagueness is acceptable.

So the emergent phase transitions are reducible: to a sufficiently rich theory T_b that takes the appropriate infinite limit; but not to a theory using finite N.

6. Other examples

There are many other examples of novel and robust behaviour in a limit. Some are related to our examples: e.g. superselection in the $N \to \infty$ limit of quantum mechanics.

Others are based on the idea of a continuous model of a fluid. Yet others involve examining wave phenomena in the limit of very short wavelengths:

1: the geometric optics limit ($\lambda \to 0$) of wave optics; and similarly

2: the classical limit ($\hbar \to 0$) of quantum mechanics; (or rather: some aspects of this limit! Which involves so much else, such as: coherent states, decoherence, the measurement problem ...)