The Angular Sum of a Triangle

G W Gibbons

Trinity Mathematical Society

Feb 4th 2008

I suppose for many of us, it was a memorable event in our lives when we were told, or discovered, that sum of the interior angles of a triangle, no matter of what shape, must add up to 180 degrees.

Later we will have been shown a proof, and possibly told that this is a classic example of the intellectual heritiage of the Greeks: imperishable truths about the world that only mathematics can provide.

However a proof is only as good as its assumptions and its utility and depends on identifying in the real world objects corresponding to the abstract entites manipulated in the proof.

In the case of the triangle law, the mathematical assumption is that the edges of the triangle are straight lines as described by Euclidean geometry and the physical assumption that these straightlines may be identified with the paths of light rays.

The crucial assumption needed to prove the triangle law is Euclid's fifth, the so-called parallel axiom. A convenient formulation is as follows

Consider an isosceles triangle ABC whose sides AC and BC are equal. As the point C recedes to infinity the angles $CAB = CBA$ tend to $\frac{\pi}{2}$, independently of the distance $AB = 2r$.

However around 1829 Lobachevsky and Bolyai realised that this assumption can be replaced by the assumption that the angle CAB, called the *angle of parallelism* $\Pi(r)$ depends on the distance r.

In fact they constructed an alternative Non-Euclidean Geometry in which the idea of congruent figures still makes sense but for which

$$\sin \Pi = \frac{1}{\cosh \frac{r}{R}} \,. \tag{1}$$

, where R is called the radius of curvature of Lobachevsky or Hyperbolic space.

For small r we obtain the Euclidean value $\frac{\pi}{2}$ but in general for Lobachevsky space the angle of parallelism is less than $\frac{\pi}{2}$.

As a consequence in Lobachevsky space the angular sum of a triangle is less than 180 degrees.

Lobachevsky

Of course the assumption that C may recede all the way to infinity can also be dropped, and this gives rise to spherical, or elliptic space in which the angular sum of a triangle is greater than 180 degrees

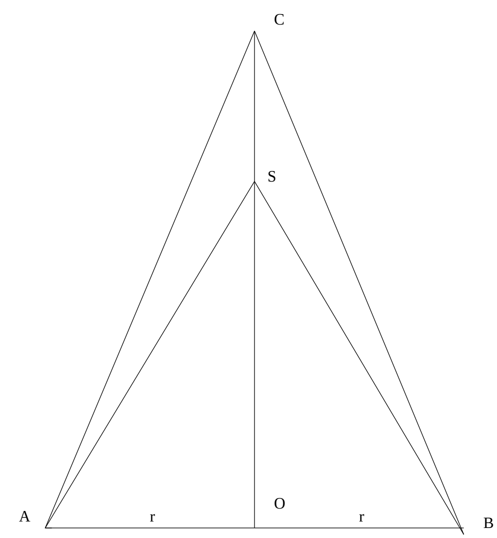
Thus we have three (simply connected) congruence geometries characterised by by a radius R and for which the angle sum of a triangle turns out to equal, up to a sign, the area of the triangle in units of R^2.

$$\angle ACB + \angle CBA + \angle BAC = \pi \pm \frac{\text{Area}(ABC)}{R^2} \tag{2}$$

Throughout the nineteenth century there were attempts by some astronomers to measure, by means of light rays, the sign and magnitude of “space” , assumed to be static and of constant curvature, by using the method of parallax.

Suppose that A and B are the positions of the earth on its orbit around the sun at times which differ by six months. If a star is situated at S somewhere on the line OC, where O is the midpoint of the side AB, and if at C there is some much more distant star, then the angles CAS, SAB and CBS and SBA may be measured. One wants, to calculate the distance 0S, the true parallax, that is the angle $p = AS0$. What one may measure, are $p_t = \frac{\pi}{2} - SA0$, by a transit circle say, and the angle $p_b = CAS$ as did Bessel by comparing the position of the star S with that of a distant star C.

Parallax geometry



Euclidean geometry implies that

$$CAS + SAB = \frac{\pi}{2} \ \Rightarrow p_t = p_b \, . \tag{3}$$

But in Hyperbolic geometry

$$p_t - p_b = \frac{\pi}{2} - \Pi(r) \approx \frac{R}{r}, \tag{4}$$

Thus a measurement of $p_t - p_b$ will allow an estimate of the radius of curvature R.

In the nineteenth century, it was customary to refer to p_t, as 'the parallax 'or more generally $\frac{1}{2}(\pi - SAO - SBO)$ was called the *annual parallax* and indeed before the work of Bessel on 61 Cygni these were the only parallaxes which were measured. Lobachevsky pointed out in 1829 that the angle p_t is *bounded below*, no matter how far away a star is, it follows from (1) that it cannot be less than

$$p_t \geq \sin^{-1}(\tanh r)\,. \tag{5}$$

The smallest measured parallax then gives a lower bound for the radius of curvature. Using (an inaccurate) value for the parallax of Sirius (1.24 as, 4 times too large) Lobachevsky obtained a lower bound of 1.66×10^5 au.

A lower bound for the measured parallax p_t may still hold if the curvature varies in space. It only requires (see later) that the integral of the curvature over triangle ASO be negative no matter how distant the star at S is.

In the case of positive curvature, one finds that the measured parallax p_t can become *negative* , and this was widely regarded as a "smoking gun" for space curvature.

It is widely believed that Gauss checked the angle sum formula as part of his geodetic duties at Göttingen using as vertices the peaks of the Brocken, Hoher Hagen and Inselberg, but this has been disputed.

In the early 1880's Ball reported that measurements then were insufficient to decide the matter.

In 1889 Calinon suggested that the radius could vary with time. By the 1890's the Pragmatic Philosopher Charles Sanders Peirce, son of the Harvard astronomer Benjamin Peirce somewhat optimistically wrote

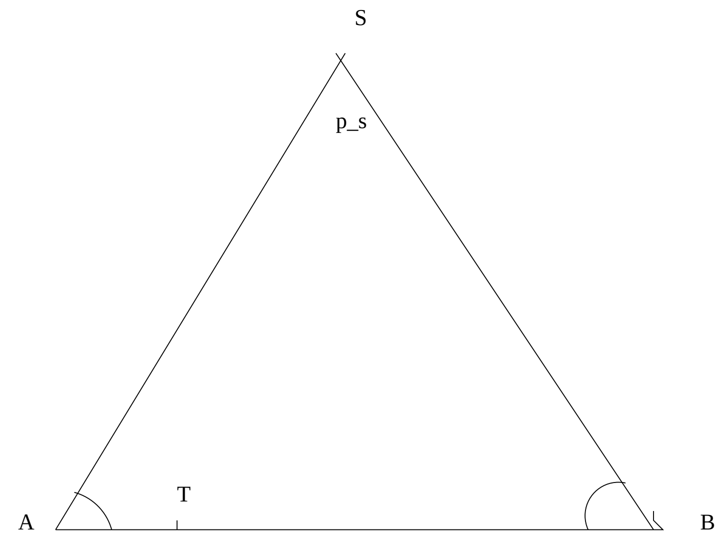
> ...our grandchildren will surely know whether the three angles of a triangle are greater or less than 180° – that they are *exactly* that amount is what nobody can ever be justified in concluding.

He then had available 40 measured parallaxes, i.e. measurements of $(\pi - SAO - SBO)$ two of which were negative. They were Aridad (α Cygni) of magnitude 1.5 and parallax $-0.082as$ and Piazzi III 422, with magnitude 7.75 and parallax measured by Ball of $-0.O45as$ Peirce ascribed these to observational error and concluded that the parallax of the furthest star measured lay in the interval $(-0.05as, 0.15as)$

In 1900 Schwarzschild, obtained the lower bound of $4 \times 10^6\,au$..

Schwarzschild looking forward to the Einstein Static Universe .

A Lower bound for the parallax.



Schwarzschild considered a star T whose direction is along the diameter AOB, and calls $p_s = \frac{1}{2}(\pi - TAS - TBS)$ the parallax. If the length of OS is d, he claims that with sufficient accuracy

$$\coth \frac{d}{R} = p_s \frac{R}{r} \,. \tag{6}$$

Thus the parallax is bounded below $p_s > \frac{r}{R}$. But at that time many stars had measured parallaxes less than .05 as, whence his lower bound for the radius of curvature.

Sommerville , writing in 1914 considered the case of α Centauri for which $p_t = 1.14\,as$ and $p_b = 0.76 \pm 0.01\,as$ giving the radius of curvature no less than $5.5 \times 10^5\,au$.

These early geometrical speculations were revolutionised in 1916 with the advent of Einstein's Theory of General Relativity. According to this theory both space and time are curved and depend on the distribution of matter. Hoever he retained the assumption that light rays follow geodesics, but now they are null-geodesics of our four-dimensional spacetime.

By 1917 Einstein produced the Einstein Static universe in which space is positively curved and of approximate radius $10^{27}\, cm$. This was essentially the same as Schwarzschild's 1900 model.

Robertson, writing in 1949, well after the advent of relativistic cosmology, reviewed Schwarzschild's work and comments that while Schwarzschild obtained lower bounds of 1600 or 64 light years depending upon whether the curvature was positive or negative respectively, by 1949 observations of galaxies stretched to over 500 million light years.

By that time, following the theoretical work of Friedman and Lemaitre, and the observational work of Hubble, it was widely accepted that the universe is not static, but rather the distance between galaxies is increasing with time.

$$ds_4^2 = -dt^2 + a^2(t) g_{ij} dx^i dx^j \tag{7}$$

where $g_{ij}dx^i dx^j$ is the 3- metric or line element of one of the three congruence geometries. This fits well with Calinon's 1889 speculation.

Actually, because the motion of light rays or null geodesics for two confomally related metrics are the same, the angular sum of a triangle in a time dependent Friedman-Lemaitre model is the same as in a static model with the “space” given by the same congruence geometry.

$$ds_4^2 = a^2(t)\{-d\eta^2 + g_{ij}dx^i dx^j\} \quad d\eta = \frac{dt}{a(t)} \tag{8}$$

Present day observations such as WMAP, using the 3 K Cosmic Microwave Background indicate that the universe is spatially flat. In other words the angle sum of an extremely large triangle is indeed 180 degrees as we were taught at elementary school, however...

In 1911 Einstein realised that the paths of light rays can become bent as they pass through gravitational fields and in 1919 Eddington showed that light from distant stars grazing the sun's disc during a solar eclipse are indeed bent by the amount that General Relativity predicts

$$\delta \approx \frac{4GM}{c^2 b} \tag{9}$$

where b is the impact parameter.

More recently astronomers have observed many gravitational lenses.

The first lensed quasar observed, Q0957+561 in 1979, has two images separated by 6.3 as. The largest observed separation so far is SDSSJ1004+4112 which has a separation of 14.62 as. Thus this effect is not small. Moreover, the two ‘dark age’galaxies with the largest observed redshifts to date, one with Z=10 and one with z=7 both owe their observation here on earth to gravitational lensing by an intervening galaxy cluster (Abell 1835 or Abell 2218 respectively) to provide the necessary magnification. More speculatively, cosmic strings might bring about comparable effects.

This suggest that the relevant curvature near a gravitating mass is positive, but this is not correct In fact we shall show later that in plane passing through the sun the relevant (Guassian) curvature is given by

$$K = -\frac{2GM}{c^2 r^3}(1 - \frac{3GM}{2c^2 r}) \tag{10}$$

which is negative.

Shortly, we shall use the Gauss-Bonnet Theorem to resolve this paradox, but before doing so we need a little more (optical) geometry.

We assume that we have a static metric

$$ds_4^2 = -|g_{00}|dt^2 + h_{ij}dx^i dx^j\,. \tag{11}$$

the spatial projections of null geodesics are, by Fermat's principle of least time, geodesics of the optical or Fermat 3-metric

$$ds_3^2 = \frac{1}{|g_{00}|} h_{ij}dx^i dx^j = f_{ij}dx^i dx^j\,. \tag{12}$$

Angles measured using the optical metric coincide with those measured using the physical metric.

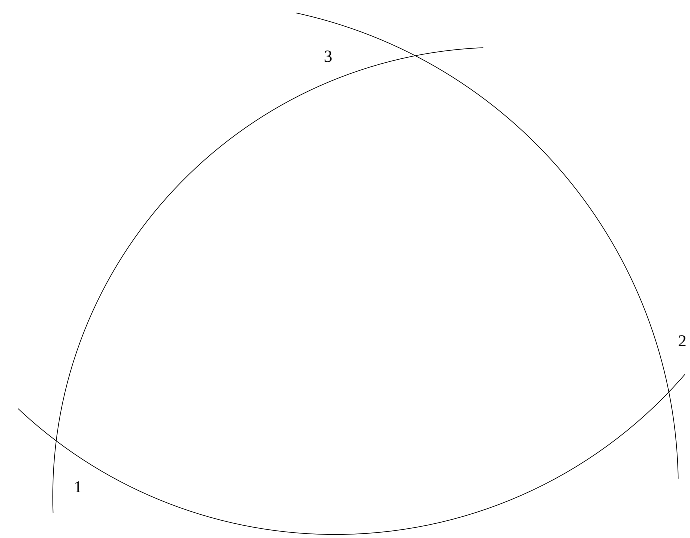
It is the curvature of the optical metric which is important for the angle sum of an optical triangle.

If $|g_{00}| = e^{2U}$ = constant, the Newtonian potential U is spatially constant as happened in the previous cases, the spatial metric and the optical metric coincude. If one likes, one may think of $|g_{00}|^{-1}$ as analogous to the refractive index of space.

Consider now geodesics lying in an oriented two-surface Σ, one may apply the Gauss-Bonnet theorem to obtain useful information about angle sums of geodesic triangles. Let $D \subset \Sigma$ be domain with Euler number $\chi(D)$ and a not necessarily connected boundary ∂D, possibly with corners at which the tangent vector of the boundary is discontinuous. If K is the Gauss curvature of D, such that $R_{ijkl} = K(f_{ik}f_{jl} - f_{il}f_{jk})$ and k the curvature of ∂D, θ_i the angle through which the tangent turns inwards at the i'th corner then

$$\boxed{\int_D K dA + \oint_{\partial D} k dl + \sum_i \theta_i = 2\pi\chi(D)\,.} \tag{13}$$

Three exterior angles



The Schwarzschild 4-metric is (with units such that G=c=1)

$$ds_4^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (14)$$

The optical 3-metric is

$$ds_3^2 = \frac{dr^2}{(1 - \frac{2M}{r})^2} + \frac{r^2}{1 - \frac{2M}{r}}(d\theta^2 + \sin^2\theta d\phi^2) \qquad (15)$$

For a single light ray we may, witout loss of generality, restrict attention to equatorial plane. For more than one geodesic, this is a simplifying assumption.

The optical metric reduces to

$$ds^2 = \frac{dr^2}{(1 - \frac{2M}{r})^2} + \frac{r^2}{(1 - \frac{2M}{r})} d\phi^2 \,. \qquad (16)$$

Note that the radial optical distance is

$$\frac{dr}{(1 - \frac{2M}{r})} = dr^\star, \qquad (17)$$

where $r^\star = r - 2M + 2M \ln(\frac{r}{2M} - 1)$ is often called the Regge-Wheeler tortoise coordinate.

There is a circular geodesic at $r = 3M$ and the horizon $r = 2M$ is at an infinite optical distance inside this at $r^\star = \infty$.

The Gauss curvature

$$K = -\frac{2M}{r^3}(1 - \frac{3M}{2r}) \tag{18}$$

is *everywhere negative*. It falls to zero like $-\frac{2M}{r^3}$ at infinity but near the horizon at which the metric has constant negative Gauss-curvature $-\frac{1}{(4M)^2}$.

The fact that the Gauss curvature is *negative* looks on the face of it rather paradoxical, since one usually thinks of gravitational fields as *focussing* a bundle of light rays. However, a spherical vacuum gravitational field does not quite act in that way. The equation of geodesic deviation governing the separation η of two neighbouring light rays in the equatorial plane is

$$\frac{d^2\eta}{dt^2} + K\eta = 0\,. \tag{19}$$

Thus neighbouring light rays actually *diverge*. The focussing effect of a gravitational lens is not, as we shall see shortly, not a local but rather a global , indeed even topological effect.

One might wonder whether the full 3-dimensional curvature ${}^3R_{ijkl}$ of the optical metric has all of its sectional curvatures negative, but this cannot be. The sectional curvature of a surface is related to the full curvature tensor by

$$ {}^3R_{ijkl} = K(f_{ik}f_{jl} - f_{1l}f_{jk}) - K_{ik}K_{jl} + K_{il}g_{jk}\,, \tag{20} $$

where K_{ij} is the second fundamental form or extrinsic curvature of the surface. For a totally geodesic surface $K_{ij} = 0$, and the two sectional curvatures agree. One such totally geodesic surface is the equatorial plane for which, as we have seen, K is negative. Another totally geodesic submanifold is the sphere at $r = 3M$ for which K is obviously positive.

The negativity of the Gauss curvature of the optical metric in the equatorial plane is a fairly *universal* property of black hole metrics. To see this we note that if

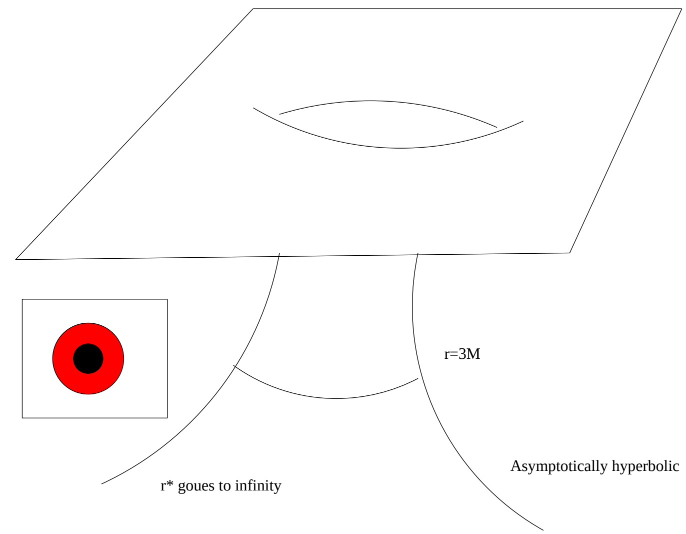
$$ds^2 = d\rho^2 + l^2(\rho) d\phi^2 \,, \tag{21}$$

then

$$K = -\frac{1}{l}\frac{d^2 l}{d\rho^2} \,. \tag{22}$$

Any metric with the same qualitative features as the Schwarzschild metric, as long as it has a *positive* mass, will have K negative. Indeed this fact might be made the basis of excluding negative mass objects observationally.

The optical wormhole



A simple calculation shows the integral over the region outside the circular geodesic at $r = 3M$ is

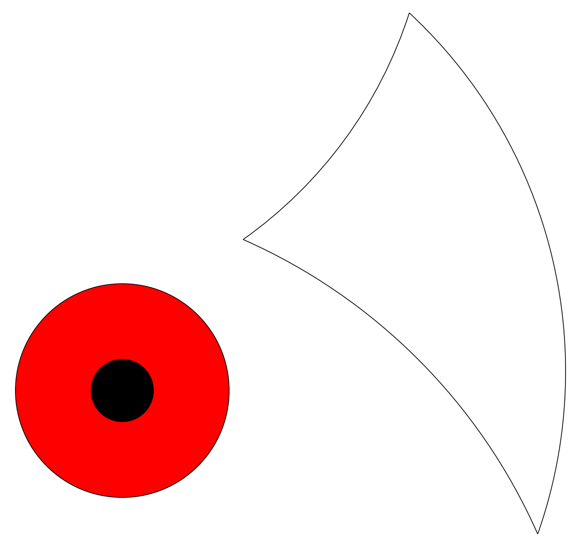
$$\int_{r \geq 3M} K dA = -2\pi \,. \tag{23}$$

Let us now apply the Gauss-Bonnet theorem to various cases.

(i) Geodesic triangle Δ not containing the the region inside $r = 3M$. In this case $\chi(\Delta) = 1$. If α, β, γ are the necessarily positive internal angles, we find that the angle sum is less that π,

$$\alpha + \beta + \gamma = \pi + \int_{\Delta} K dA < \pi \,. \tag{24}$$

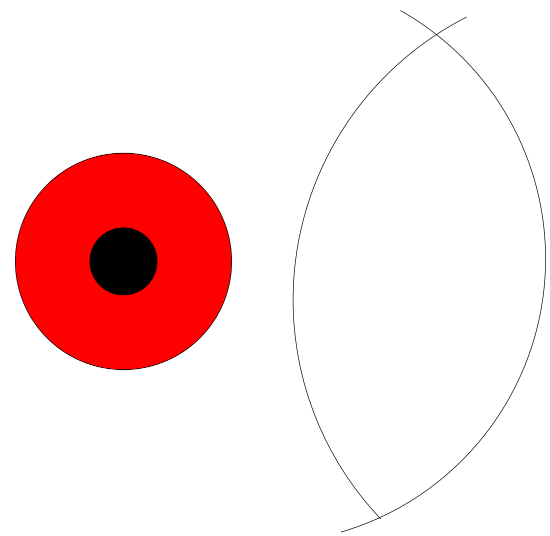
The angle sum is less than 180 degrees



(ii) Geodesic di-gon S not containing the the region inside $r = 3M$. In this case $\chi(S) = 1$. If α and β are the internal angles,

$$\alpha + \beta = \int_S K dA < 0 \,. \tag{25}$$

In other words two such geodesics cannot intersect twice if the hole is not inside the di-gon. Neither, in these circumstances, can a geodesic intersect itself because



This is *not* allowed

(iii) Geodesic loop T not containing the the region inside $r = 3M$. In this case $\chi(T) = 1$ and one finds that if if α is the internal angles,the

$$\alpha = -\pi + \int_T KdA < 0\,, \tag{26}$$

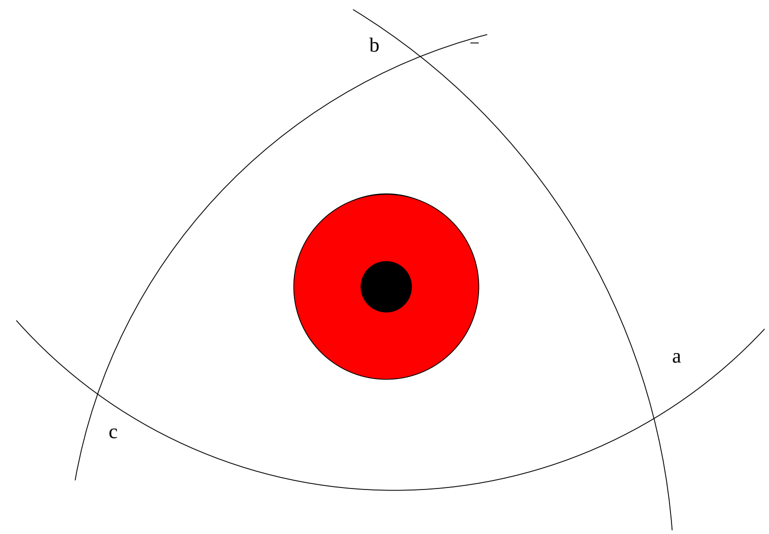
which is plainly impossible.

This might seem counter-intuitive in the light of one's usual intuition about light bending, but this feeling is dispelled by considering cases in which the domain D has two boundary components, the second, inner, one being the circular geodesic at $r = 3M$. The domain with the circle removed has the topology of an annulus and thus its Euler number vanishes.

(iv) Geodesic triangle with hole Δ_o enclosing the geodesic circle at $r = 3M$ and containing the with the region the region inside $r = 3M$ removed.

If α, β, γ are the internal angles, we find that the angle sum is greater than π,

$$\alpha + \beta + \gamma = 3\pi + \int_{\Delta_0} KdA < \pi \geq \pi \,. \tag{27}$$



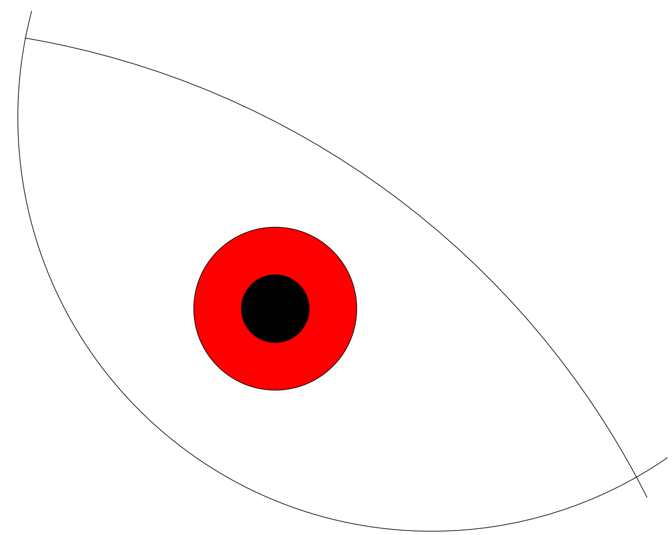
The angle sum exceeds 180 degrees

Similarly

(v) Geodesic di-gon S_0 n containing the the region inside $r = 3M$. In this case $\chi(S_0) = 0$ and one finds that if If α and β are the internal angles, then

$$\alpha + \beta = 2\pi + \int_{S_0} K dA > 0 \,. \tag{28}$$

In other words two such geodesics may intersect twice if the hole is inside the di-gon. Moreover, in these circumstances, a geodesic can intersect itself because



This *is* allowed

(vi) Geodesic loop T_0 containing the the region inside $r = 3M$. In this case $\chi(T_0) = 1$ and if If α is the internal angle, we find that

$$\alpha = \pi + \int_{T_0} K dA \,, \tag{29}$$

which is plainly possible.

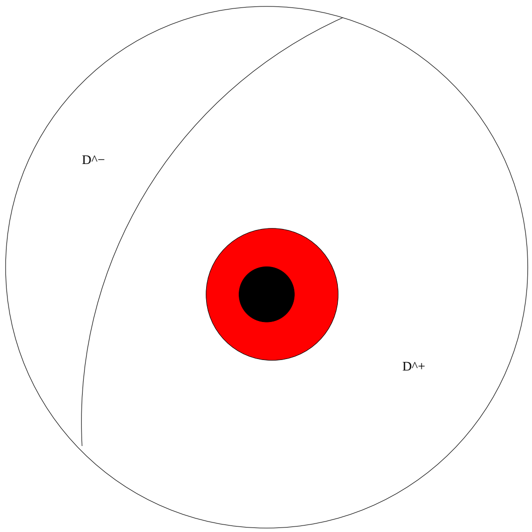
Similar results may be obtained by considering geodesics inside $r = 3M$, but now domain must not contain the horizon, otherwise $\int_D K dA$ will diverge. Near the horizon the geometry is that of Lobachebsky space with constant curvature $-\frac{1}{4M}$.

In fact this is a general feature of the optical geometry in the neighbourhood of a non-degenerate static event horizon.

(vii) Deflection. We consider a geodesic line with no self-intersection which at large distances, is radial. The angle between the asymptotes is δ, with the convention that it is positive if the light ray is bent towards the hole. The geodesic decomposes the region inside two circles, one of very large radius and the other at $r = 3M$ into two domains $D_{\pm}$ whose common boundary component consists of the geodesic, which intersects the circle at infinity at right angles. We chose D_{+} to enclose the hole so it has an inner boundary component at $r = 3M$ and a portion of the circle at infinity through which the angle ϕ has range $\pi - \delta$. Clearly D_{+} is topologically an annulus and so it has vanishing Euler number, $\chi(D_{+}) = 0$. The other domain has Euler number $\chi(D_{-}) = 1$, and ϕ ranges through $\pi + \delta$. The Gauss-Bonnet formula applied to $D_{\pm}$ acquires a contribution from the two corners and the circle at infinity. The result is

$$\delta = -\int_{D_{-}} K dA > 0. \qquad (30)$$

Light bending

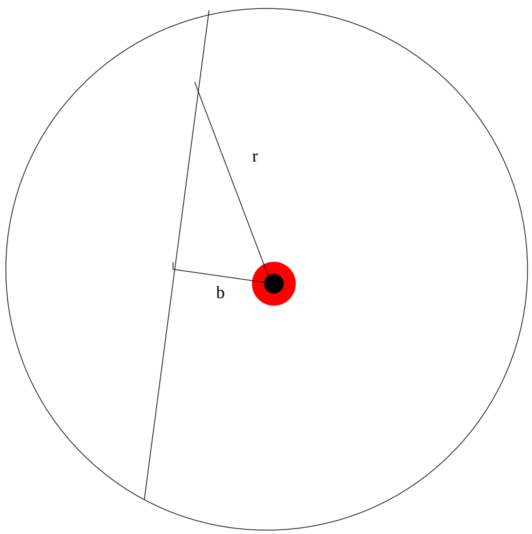


For a geodesic whose distance of closest approach is very large, we may estimate this integral by approximating the geodesic as the straight line $r = \frac{b}{\sin\phi}$. The impact parameter to this lowest non-trivial order coincides with the distance of nearest approach and equals b. To the necessary accuracy

$$KdA \approx -\frac{2M}{r^3} r dr d\phi \,. \tag{31}$$

The domain of integration D_- is, with sufficient accuracy over $r \geq \frac{b}{\sin \phi}$, $0 \leq \phi \leq 2\pi$. A simple calculation gives the classic result

$$\delta = \frac{4M}{b}. \tag{32}$$



Approximate light deflection

The same method works for any black hole metric, not just Schwarzschild, and shows that the Gauss-Bonnet method does not just give qualitative results, but it can be made into a quantitative tool.

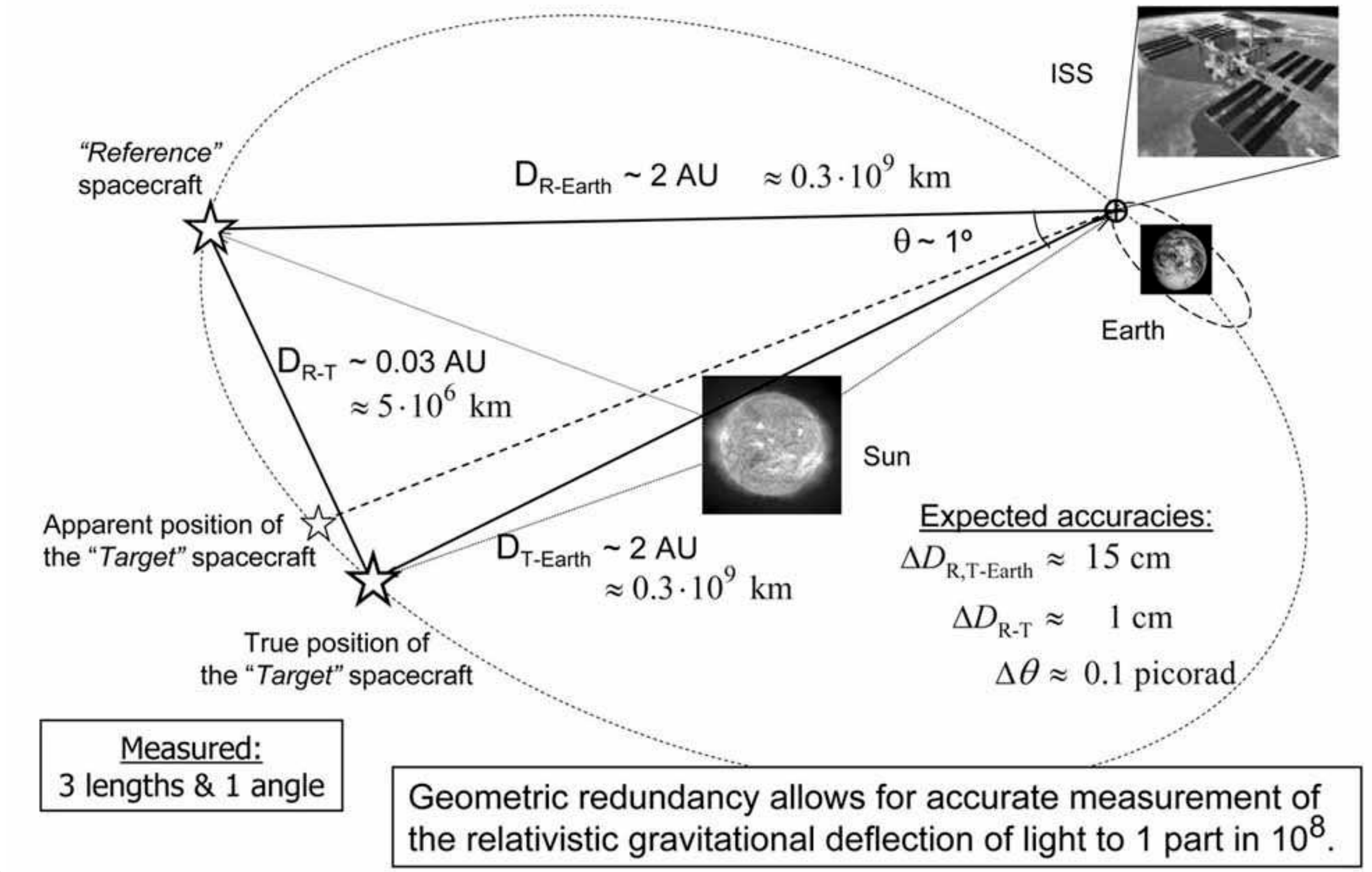
Some future missions may make these and analogous calculations of more than pruely academic interest.

The fothcoming GAIA mission, is the next step beyond HIPPARCOS, and will attain accuracies of the order of pico-radians. For parallax measurements median errors of $4\,\mu as$ for 10'th magnitude objects and $160\,\mu as$ for those at 20'th magnitude are expected

The satellite will orbit at the second Lagrange point L2 and with these accuracies the gravitational deflection due to planets may well be detectable..

Gravitational light deflection measurements are also the aim of the proposed LATOR Mission in which two satellites and the space station will form the vertices of a triangle. Three edge-lengths, a, b, c will be measured by laser timing and the angle A between light rays coming from the two satellites and arriving at the space station will also be measured.

The LATOR Mission: Relativistic Deflection of Light



If spacetime were flat these quantities would satisfy the usual cosine formula. In the presence of curvature one gets a different relation. For example if the curvature was constant and negative, one would get the hyperbolic version of the formula of Albategnius

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A \,, \tag{33}$$

where a, b, c are measured in curvature units.

Al-Battani

Of course the curvature K is not constant , but it is negative. In fact we shall show later that in plane passing through the sun the relevant curvature is

$$K = -\frac{2M}{r^3}(1 - \frac{3M}{2r}) \tag{34}$$

and can be be made the basis of an approximate formula for the size of the effect that LATOR is intended to measure.

However, despite the fact that it is an elementary exercise to reduce to quadratures the motion of light rays in the Schwarzschild metric, it would seem to be an extremely challenging task to produce an analytic formula for analogue of the cosine formula in general. An approximate expression for the angular sum of a large triangle, or indeed any polygon will be given later.

The accuracy anticipated for LATOR would allow a measurement of the *second order* light deflection by the sun. This is 3.5 μas.

I will conclude by mentioning some outstanding, and, in principle, elementary problems involving light triangles near a black hole.

Post-Newtonian Angular sum of a large polygon surrounding the sun.

The Gauss-Bonnet theorem tells one that a geodesic n-gon with internal angles α_i, $i = 1\,2, \dots n$ enclosing the hole has angular sum

$$\sum_i \alpha_i = (n-2) - \int_{D'} K dA \,, \tag{35}$$

the integral being taken over the region D' *outside* the triangle. We perform the integral by dealing with each of the n sectors S_i subtended by the sides separately and then adding the results since

$$D' = \cup_i S_i \,. \tag{36}$$

We choose coordinates so that the radial lines $\phi = \phi_i$ and $\phi = -\phi'_i$ pass through the ends of the sides. If the geodesic polygon is large, one may approximate its i'th side by a straight line

$$r = \frac{b_i}{\cos\phi}, \qquad -\phi_i \leq \phi \leq \phi_i', \tag{37}$$

with $0 \leq \phi_i, \phi_i' \leq \pi$ and where b_i is its distance of closest approach to the origin. The integral over the i'the sector S_i may be evaluated as was done above for the scattering and one finds

$$\int_{S_i} K dA = -\frac{2M}{b_i}(\sin\frac{\phi_i}{2} + \sin\frac{\phi_i'}{2}). \tag{38}$$

Adding the contributions from the n-sectors gives

$$\boxed{\sum_i \alpha_i = (n-2)\pi + 2M\sum_i(\sin\frac{\phi_i}{2} + \sin\frac{\phi_i'}{2})\frac{1}{b_i}.} \tag{39}$$

For example, for $n = 3$, the angle sum is indeed greater than π. The

n quantities ϕ_i, ϕ'_i, b_i are of course not independent. For example

$$\sum_i (\phi_i + \phi'_i) = 2\pi. \tag{40}$$

Post-Newtonian Angle sum of a triangle not enclosing the sun

Suppose that side 3 is closest to the sun, such that sides 1 and 3 intersect at a vertex point on the other side of the 3rd side from the sun. Suppose, as before, that S_i is the part of the sector subtended by the i'th side which lies beyond the i'th side.

The angle sum will be

$$\boxed{\alpha + \beta + \gamma = \pi + \int_{S_3} K dA - \int_{S_1} K dA - \int_{S_2} K dA\,.} \tag{41}$$

Each of the three integrals in (41) is given by an expression of the form (38).

Exact Analytic Results

If $u = \frac{1}{r}$, the orbit satisfies

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2 \Rightarrow (\frac{du}{d\phi})^2 = b^{-2} - u^2 + 2Mu^3 - 2Mb^{-3}, \tag{42}$$

where b is the radius of nearest approach. The angle Θ between the orbit and the radial direction is given by

$$\tan\Theta = \frac{1}{r}\frac{dr}{d\phi}\frac{1}{\sqrt{1 - \frac{2M}{r}}}. \tag{43}$$

Thus

$$\tan\Theta = \frac{1}{\sqrt{1 - \frac{2M}{r}}}\sqrt{\frac{r^2}{b^2} - 1 - 2\frac{M}{r}(\frac{r^3}{b^3} - 1)}. \tag{44}$$

Note that, as defined, Θ is positive and vanishes at a turning point. Now we may evaluate (44) at the two ends of the i'th geodesic side. Thus

$$\tan\Theta_i = \frac{1}{\sqrt{1-\frac{2M}{r_i}}}\sqrt{\frac{r_i^2}{b_i^2} - 1 - \frac{M}{r_i}(\frac{r_i^3}{b_i^3} - 1)}\,, \tag{45}$$

$$\tan\Theta_i' = \frac{1}{\sqrt{1-\frac{2M}{r_i'}}}\sqrt{\frac{(r_i')^2}{b_i^2} - 1 - 2\frac{M}{r_i'}(\frac{(r_i')^3}{b_i^3} - 1)}\,. \tag{46}$$

Now the angular sum is

$$n\pi - \sum_i \Theta_i + \Theta_{i+1}'\,. \tag{47}$$

Of course not all the radial radii are independent. Obviously

$$r_i = r'_{i+1}, \qquad \text{etc.} \tag{48}$$

The optical lengths of the sides t_i , as would be measured by laser timing, and the angles ϕ_i and ϕ'_i are given by integrals which, in general, can only be expressed in terms of elliptic functions. Thus

$$\phi_i = \int_{b_i}^{r_i} \frac{dr}{r^2} \frac{1}{\sqrt{b_i^{-2} - r^{-2} + 2Mr^{-3} - 2Mb_i^{-3}}} \tag{49}$$

$$\phi'_i = \int_{b_i}^{r'_i} \frac{dr}{r^2} \frac{1}{\sqrt{b_i^{-2} - r^{-2} + 2Mr^{-3} - 2Mb_i^{-3}}} \tag{50}$$

and

$$t_i = \int_{b_i}^{r_i} \frac{dr}{(1-\frac{2M}{r})b_i\sqrt{1-\frac{2M}{b_i}}} \frac{1}{\sqrt{b_i^{-2} - r^{-2} + 2Mr^{-3} - 2Mb_i^{-3}}}, \quad (51)$$

$$+ \int_{b_i}^{r'_i} \frac{dr}{(1-\frac{2M}{r})b_i\sqrt{1-\frac{2M}{b_i}}} \frac{1}{\sqrt{b_i^{-2} - r^{-2} + 2Mr^{-3} - 2Mb_i^{-3}}}. \quad (52)$$

In the Post-Newtonian approximation, this formula should reduce to the Shapiro time delay formula. Evidently however, the various quantities are only rather implicitly related *. Unfortunately, unless there are some miraculous relations among elliptic functions, there is no obvious way of getting out less implicit relations, even just to Post-Post-Newtonian order, as required by the LATOR project.

*the orbits are given by elliptic functions

Elliptic functions

Geodesics in the Schwarzschild metric may be integrated using elliptic functions. In the case of null geodesics one has

$$\frac{M}{2r} - \frac{1}{12} = p(\phi + \text{constant}), \tag{53}$$

where Weirstrass's function p satisfies

$$p'^2 = 4p^3 - g_2 p - g_3 \,, \tag{54}$$

with

$$g_2 = \frac{1}{12}\,, \qquad g_3 = \frac{1}{216} - \frac{M^2}{4b^2}(1 - \frac{2M}{b})\,. \tag{55}$$

Exact solutions

One may check that

$$u = \frac{1}{3M} - \frac{1}{M}\frac{1}{\cosh\phi + 1} \tag{56}$$

and

$$u = \frac{1}{M}\frac{1}{\cos\phi + 1} \tag{57}$$

are exact solutions. The former(56) starts at infinity at $\phi = \cosh^{-1}(2)$ and moves inwards, spirally around the circualr orbit at $r = 3M$.

In fact (56) is not a Weierstrass function but if one adds $i\frac{\pi}{2}$ to the argument ϕ one gets

$$u = \frac{1}{3M} + \frac{1}{M}\frac{1}{\cosh\phi - 1}\,, \tag{58}$$

which is a Weierstrass function. This orbit starts from $r = 0$ at $\phi = 0$ and moves outwards ultimately endlessly approaching the circle at $r = 3M$.

Addition Formulae The *addition formula* for Weierstrass elliptic functions states that (see Whittaker and Watson page 441)

$$4(p(x) + p(y) + p(x+y)) = (\frac{p'(x) - p'(y)}{p(x) - p(y)})^2 . \tag{59}$$

Thus

$$(\frac{u'(\phi_1) - u'(\phi_2)}{u(\phi_1) - u(\phi_2)})^2 = 2M(u(\phi_1) + u(\phi_2) + u(\phi_1 + \phi_2)) - 1 . \tag{60}$$

Define an angle ψ by

$$\tan \psi = \frac{dr}{r d\phi} . \tag{61}$$

Clearly ψ is the complement of the angle between the curve and the radial direction *if we use Euclidean geometry* rather than the correct geometry. Suppose $r_1 = r(\phi_1), r_{12} = r(\phi_+\phi_2)$ etc, then

$$(r_2 \tan\psi_1 - r_1 \tan\psi_2)^2 = (r_1 - r_2)^2 (2M(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_{12}}) - 1) \quad (62)$$

For thus orbits which allow u to get very small, for example those which always remain outside $r = 6M$, the right handside of (62) is negative and so they cannot be described by a real Weierstrass function of a real argument. For those orbits which which always lie inside $r = 3M$, the right handside of (62) is positive, and they do satisfy the addition formula. In particular this applies to the simple explict solution (58).

Conclusions The question: what is the angle sum of triangle turns out to be more involved than it at first appears. At the scale of the universe as a whole, space is Euclidean, and the answer is 180 degrees after all. Near gravitating bodies for a triangle which does not enclose the gravitating centre it is *less than* 180 degrees but for a triangle which encloses the gravitating centre it is *greater* than 180 degrees despite the fact that the (optical) curvature is everwhere *negative*. The resolution of this paradox involves a simple topological fact.

These seemingly purely academic questions have important implicatins for some upcoming and projected space missions.