Twistor Transform

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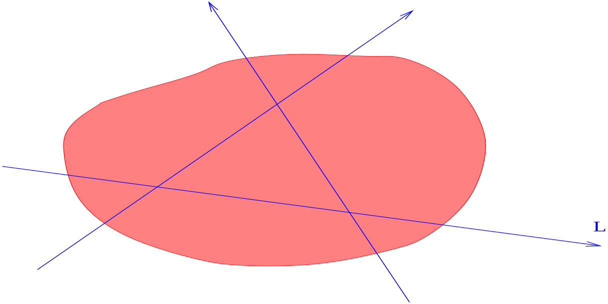
- 1917 Radon. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with decay condition ot ∞, $L \subset \mathbb{R}^2$ oriented line.

$$\phi(L) := \int_L f.$$

There exist an inversion formula $\phi \longrightarrow f$.



Differential Equations

- 1938 Fritz John. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, oriented line $L \subset \mathbb{R}^3$.
  Define $\phi(L) = \int_L f$, or

$$\phi(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2, s) ds.$$

- The space of oriented lines is 4 dimensional, and $4 > 3$ so expect one condition on ϕ.
- Differentiate under the integral: ultrahyperbolic wave equation

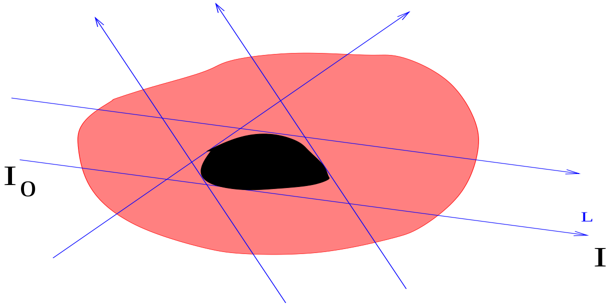
$$\frac{\partial^2 \phi}{\partial \alpha_1 \partial \beta_2} - \frac{\partial^2 \phi}{\partial \alpha_2 \partial \beta_1} = 0.$$

- Change coordinates $\alpha_1 = x + y, \alpha_2 = t + z, \beta_1 = t - z, \beta_2 = x - y$.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial t^2} = 0.$$

  Relevant to physics with two times!

- 1963 Cormack. Hole theorem.



$$\phi(L) = \int_L \frac{dI}{I} = \log I - \log I_0 = -\int_L f$$

1979 Nobel Prize (in medicine) for image reconstruction.

Two more integral formulae

- 1904 Bateman (Prize Fellow at Trinity) Laplace equation in $\mathbb{R}^4$

$$\phi(x, y, z, t) = \oint_{\Gamma \subset \mathbb{CP}^1} f((z + i\tau) + (x + iy)\lambda, (x - iy) - (z - i\tau)\lambda, \lambda) d\lambda$$

verify $$\frac{\partial^2 \phi}{\partial \tau^2} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

- 1967 Penrose (Twistor theory). Wave equation in Minkowski space $\mathbb{R}^{3,1}$.

$$\phi(x, y, z, t) = \oint_{\Gamma \subset \mathbb{CP}^1} f((z + t) + (x + iy)\lambda, (x - iy) - (z - t)\lambda, \lambda) d\lambda$$

verify $$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0.$$

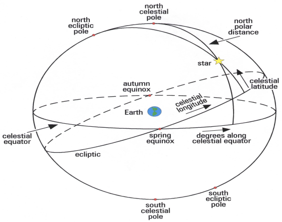
- Mathematically sophisticated: Could modify a contour and add a holomorphic function inside the contour to f. Needs sheaf cohomology (Atiyah. Master of Trinity 1990–1997).

Complex Numbers in Physics

- Quantum Physics. Complex wave function, Hilbert spaces, ...
- Classical Physics. Complex numbers in the sky! Celestial sphere

$(u_1)^2 + (u_2)^2 + (u_3)^2 = 1$

Stereographic projection onto a plane



- From north pole $(0,0,1)$, $\lambda = \frac{u_1 + iu_2}{1 - u_3}$.
  From south pole $(0,0,-1)$, $\tilde{\lambda} = \frac{u_1 - iu_2}{1 + u_3}$.
- On the overlap $\tilde{\lambda} = 1/\lambda$. This makes S^2 into a complex manifold $\mathbb{CP}^1$ (Riemann sphere).
- Möbius transformations $\xrightarrow{2:1}$ Lorentz transformations.

Twistor Programme

- Twistor correspondence

$$
\begin{aligned}
\text{Space time} &\longleftrightarrow \text{Twistor space} \\
\text{Point} &\longleftrightarrow \text{Complex line } \mathbb{CP}^1 \\
\text{Light ray} &\longleftrightarrow \text{Point.}
\end{aligned}
$$

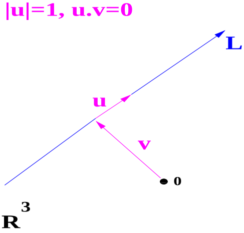
- Space-time points are derived objects in twistor theory. They become 'fuzzy' after quantisation. Attractive framework for quantum gravity.
- 40 years of research: No major impact on physics (so far). Surprisingly major impact on pure mathematics: representation theory, differential geometry, solitons, instantons, ...

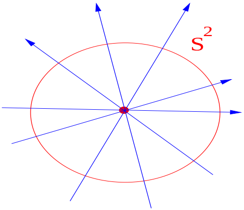
Differential equations and complex numbers.

- Harmonic functions on $\mathbb{R}^2$. Complex numbers $\mathbb{R}^2 = \mathbb{C}$.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad \phi = \mathrm{Re}(f(\zeta)), \quad \zeta = x + iy.$$

- Harmonic functions on $\mathbb{R}^3$? Problem: 3 is an odd number, $\mathbb{R}^3 \neq \mathbb{C}^n$.
- Twistor space $\mathbb{T}$ = space of oriented lines in $\mathbb{R}^3$. Line $\mathbf{v} + t\mathbf{u}, \quad t \in \mathbb{R}$.

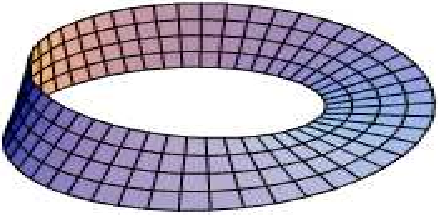




Dimension of $\mathbb{T}$ is four (even!).

Oriented Lines in $\mathbb{R}^3$

- $\mathbb{T} = \{(\mathbf{u}, \mathbf{v}) \in S^2 \times \mathbb{R}^3,\ \mathbf{u}.\mathbf{v} = 0\}$. For each fixed $\mathbf{u}$ this space restricts to a tangent plane to S^2. The twistor space is the union of all tangent planes – the tangent bundle TS^2.
- Topologically nontrivial: Locally $S^2 \times \mathbb{R}^2$ but globally twisted



- Reversing the orientation of lines $\tau : \mathbb{T} \longrightarrow \mathbb{T}$, $\tau(\mathbf{u}, \mathbf{v}) = (-\mathbf{u}, \mathbf{v})$.
- Points $\mathbf{p} = (x, y, z)$ in $\mathbb{R}^3$ two–spheres in $\mathbb{T}$; τ–invariant maps

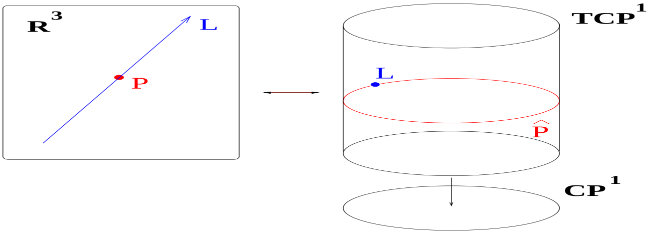
$$\mathbf{u} \longrightarrow (\mathbf{u}, \mathbf{v}(\mathbf{u}) = \mathbf{p} - (\mathbf{p}.\mathbf{u})\mathbf{u}) \in \mathbb{T}.$$

- Holomorphic coordinates

$$\lambda = \frac{u_1 + iu_2}{1 - u_3} \in \mathbb{CP}^1 = S^2, \quad \eta = \frac{v_1 + iv_2}{1 - u_3} + \frac{u_1 + iu_2}{(1 - u_3)^2} v_3.$$

  Need another coordinate patch $(\tilde{\lambda}, \tilde{\eta})$ containing $\mathbf{u} = (0, 0, 1)$. On the overlap $\tilde{\lambda} = 1/\lambda, \tilde{\eta} = -\eta/\lambda^2$.
- Points in $\mathbb{R}^3$ are τ-invariant holomorphic maps $\mathbb{CP}^1 \to T\mathbb{CP}^1$

$$\lambda \to (\lambda, \eta = (x + iy) + 2\lambda z - \lambda^2(x - iy)).$$



HARMONIC FUNCTIONS ON $\mathbb{R}^3$

To find a harmonic function at $P = (x, y, z)$:

- Restrict a twistor function $f(\lambda, \eta)$ to $\hat{P} = \mathbb{CP}^1 = S^2$.
- Integrate along a closed contour

$$\phi(x, y, z) = \oint_{\Gamma \subset \hat{P}} f(\lambda, (x + iy) + 2\lambda z - \lambda^2 (x - iy)) d\lambda,$$

- Differentiate under the integral to verify

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

- This formula was given in 1903 by Whittaker, yet another Fellow of Trinity.

Magnetic monopole

- Dirac. Are there point magnetic charges? If yes, quantisation of electric charge e can be explained

$$eg = 2\pi N, \qquad N = 1, 2, 3, \ldots \quad .$$

- Maxwell equations $=$ $U(1)$ gauge theory. Nonabelian gauge theory: Replace $U(1)$ by a non-abelian group $SU(2)$. (Relevant in elecro–weak model).
- $(A_j(\mathbf{x}), \Phi(\mathbf{x}))$ anti-hermitian 2x2 matrices on $\mathbb{R}^3$.
- Nonabelian magnetic field

$$F_{jk} = \frac{\partial A_j}{\partial x^k} - \frac{\partial A_k}{\partial x^j} + [A_j, A_k], \quad j, k = 1, 2, 3.$$

- Monopole equation

$$\frac{\partial \Phi}{\partial x^j} + [A_j, \Phi] = \frac{1}{2}\varepsilon_{jkl}F_{kl}.$$

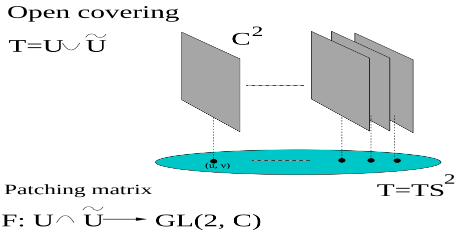
System of non-linear PDEs.

Twistor solution to the monopole equation

- Given $(A_j(\mathbf{x}), \Phi(\mathbf{x}))$ solve a matrix ODE along each oriented line $\mathbf{x}(t) = \mathbf{v} + t\mathbf{u}$

$$\frac{dV}{dt} + (u^j A_j + i\Phi)V = 0.$$

  Space of solutions at $p \in \mathbb{R}^3$ is a complex vector space $\mathbb{C}^2$.
- Complex vector bundle over $\mathbb{T}$ with patching matrix $F(\lambda, \overline{\lambda}, \eta, \overline{\eta})$.



- Monopole equation $\longleftrightarrow$ Cauchy–Riemann eq. $\frac{\partial F}{\partial \overline{\lambda}} = 0, \frac{\partial F}{\partial \overline{\eta}} = 0.$
- Holomorphic vector bundles over $T\mathbb{CP}^1$ - well understood in algebraic geometry. Take one and work backwards to construct a monopole.

BPS monopole

- Boundary conditions

$$-\frac{1}{2}\text{Trace}\,(\Phi^2) \longrightarrow 1 - \frac{n}{r} + O(r^{-2}) \quad \text{as} \quad r \longrightarrow \infty$$

- n is a monopole number. It is a topological (homotopy) invariant $n = [\Phi_\infty]$ of

$$\Phi_\infty : S^2 \longrightarrow S^2.$$

(Analogous to a winding number of a smooth map $S^1 \longrightarrow S^1$).

- Prasad–Sommerfield one-monopole solution

$$\Phi = \frac{i}{2}\frac{x_j}{r}\Big(\coth(r) - \frac{1}{r}\Big)\sigma_j, \qquad A_j = -\frac{i}{2}\varepsilon_{jkl}\frac{x_j}{r^2}\Big(1 - \frac{r}{\sinh{(r)}}\Big)\sigma_l.$$

Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Non-local construction with roots in the 19th century Klein correspondence (projective geometry).

$$\text{point} \quad \longleftrightarrow \quad \text{line (complex)}$$
$$\text{line} \quad \longleftrightarrow \quad \text{point.}$$

- Complex numbers are essential

$$\text{nonlinear PDEs} \quad \longleftrightarrow \quad \text{linear Cauchy–Riemann equations.}$$

- Its status as a physical theory is not clear
  - Weakness: effective in lower dimensions (3 or 4) and not string theoretic 10 or 11.
  - Strength: effective in lower dimensions (3 or 4) and not string theoretic 10 or 11.